Characterization of topological quantum phase transitions in the Kitaev model

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References: Phys. Rev. Lett. **98**, 087204 (2007);
Preprint, cond-mat/07053499.

Outline

- Introduction to topological quantum computation and the Kitaev model
- Jordan-Wigner transformation and a novel Majorana fermion representation of spin-1/2 operators
- Topological continuous quantum phase transitions
- Non-local string order parameters from the duality transformation of the spins
- Topological excitations of the Kitaev model
- Conclusions

Contrast Classical and Quantum Bits

Classical bit has two states :

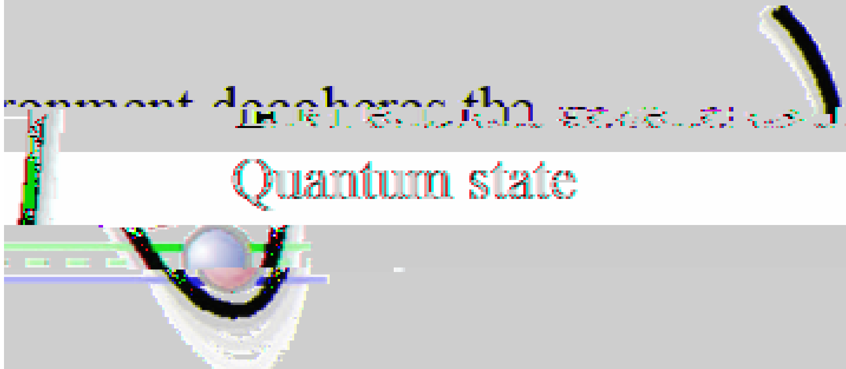
$|0>$ and $|1>$

Quantum bit is described by the state :

$|\psi_0>=\alpha_0|0>+\alpha_1|1>$

ronment decoheres the

Quantum state



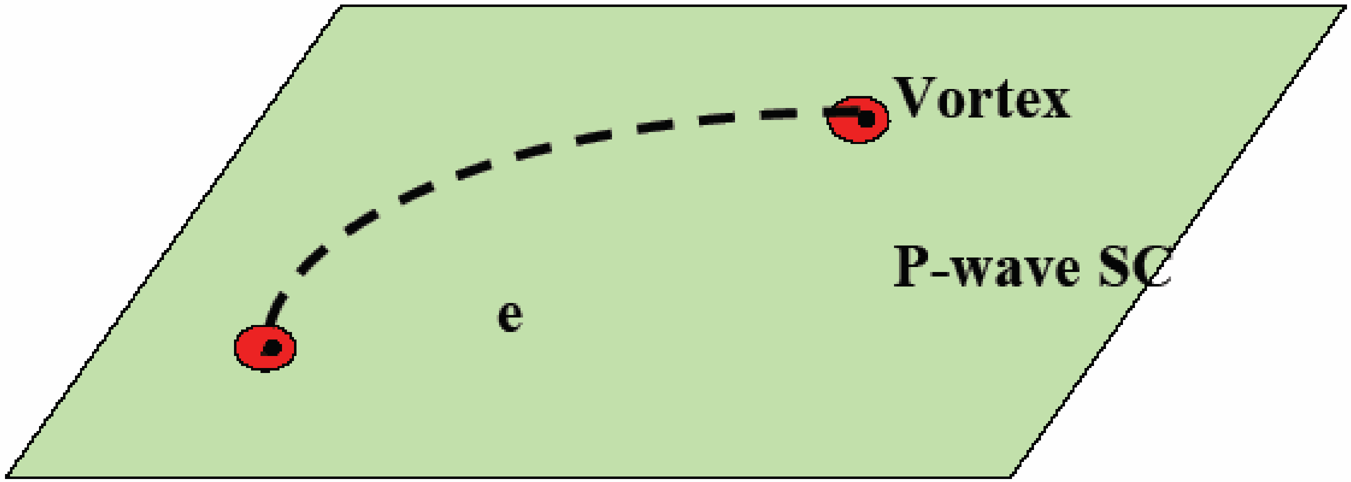
Envi

y to topology

TQC: Look for a quantum state sensitive only

(Spin polarized electron systems)

Non-Local Occupation of an Electron



It takes a *pair* of quantum states to accommodate an electron!

Non-locality

Statistics

What happens to a many-particle wavefunction under exchange of identical particles

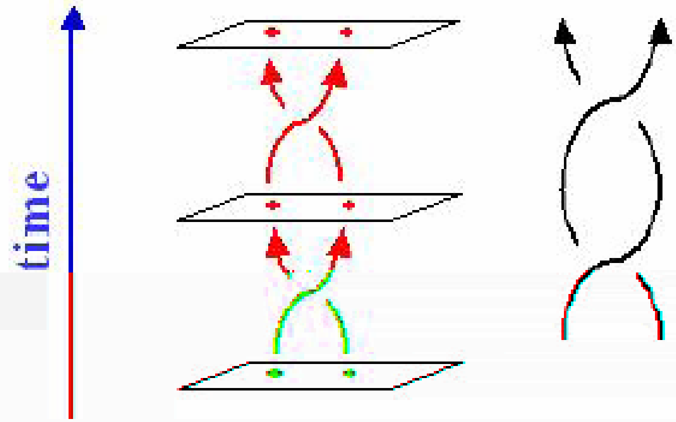
Naive Expectation

Exchanging twice should be identity

Bosons : $\psi(r_1, r_2, r_i) = \psi(r_2, r_1, r_i)$

Fermions : $\psi(r_1, r_2, r_i) = -\psi(r_2, r_1, r_i)$

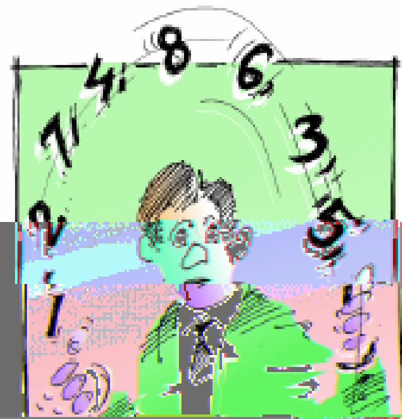
In 2+1 Dimensions: Two Exchanges $\neq$ Identity



In 2+1 Dimensions: Two Exchanges $\neq$ Identity

No Knots in World Lines in 3+1 D !

Statistics



Bosons : $\psi(r_1, r_2, r_i) = \psi(r_2, r_1, r_i)$

Fermions : $\psi(r_1, r_2, r_i) = -\psi(r_2, r_1, r_i)$

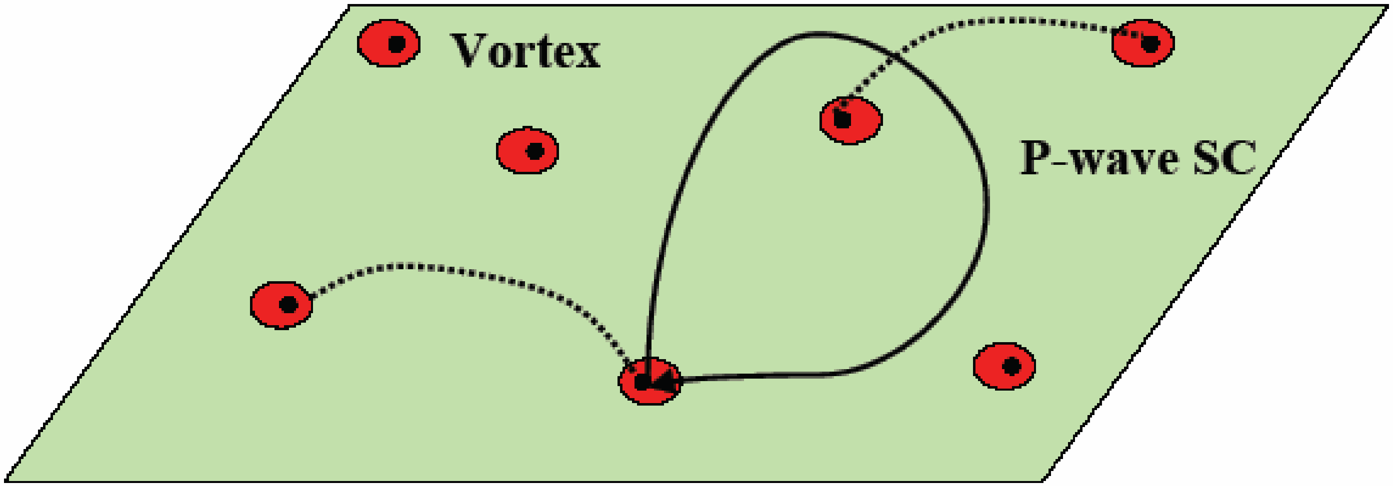
Anyons (2D) : $\psi(r_1, r_2, r_i) = e^{i\theta}\psi(r_2, r_1, r_i)$

Non-Abelian Anyons(2D): $\boxed{\psi_j(r_1, r_2, r_i) = M_{jk}\psi_k(r_2, r_1, r_i)}$

(Degenerate states)

Statistics can be non-Abelian!

Non-Abelian Statistics



Degenerate set of ground states

$$\psi_j(r_1, r_2, r_i) = M_{jk}\psi_k(r_2, r_1, r_i)$$

Non-Locality + Non-Abelian Statistics

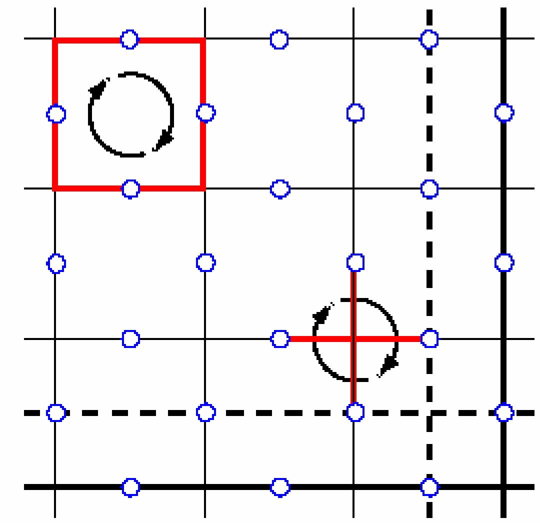
by Topology Many-particle quantum state is manipulable only

Fault-tolerant quantum computation in a toric code model

A. Kitaev, Ann Phys 303, 2 (2003); cond-mat/9707021

$$H = -J_e \sum_{\text{vertices}} A_s - J_m \sum_{\text{plaquettes}} B_p,$$

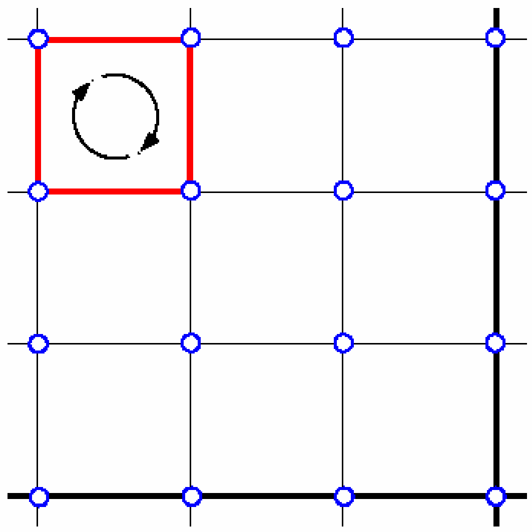
$$A_s = \prod_{j \in \text{star}(s)} \sigma_j^x, \quad B_p = \prod_{j \in \text{boundary}(p)} \sigma_j^z.$$



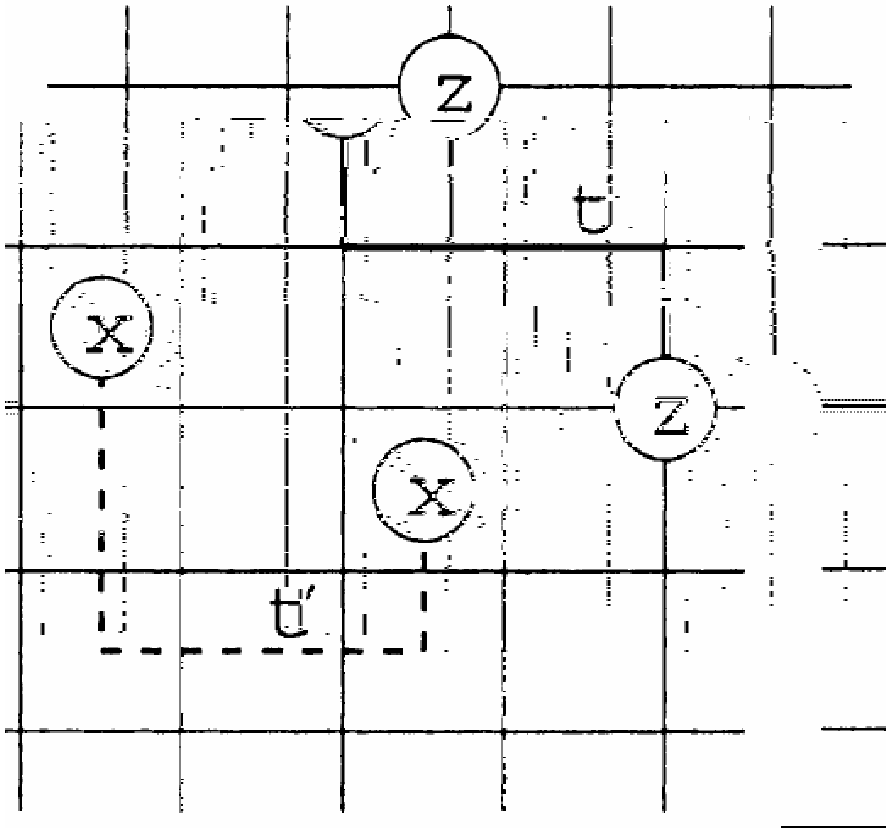
Wen's plaquette model:

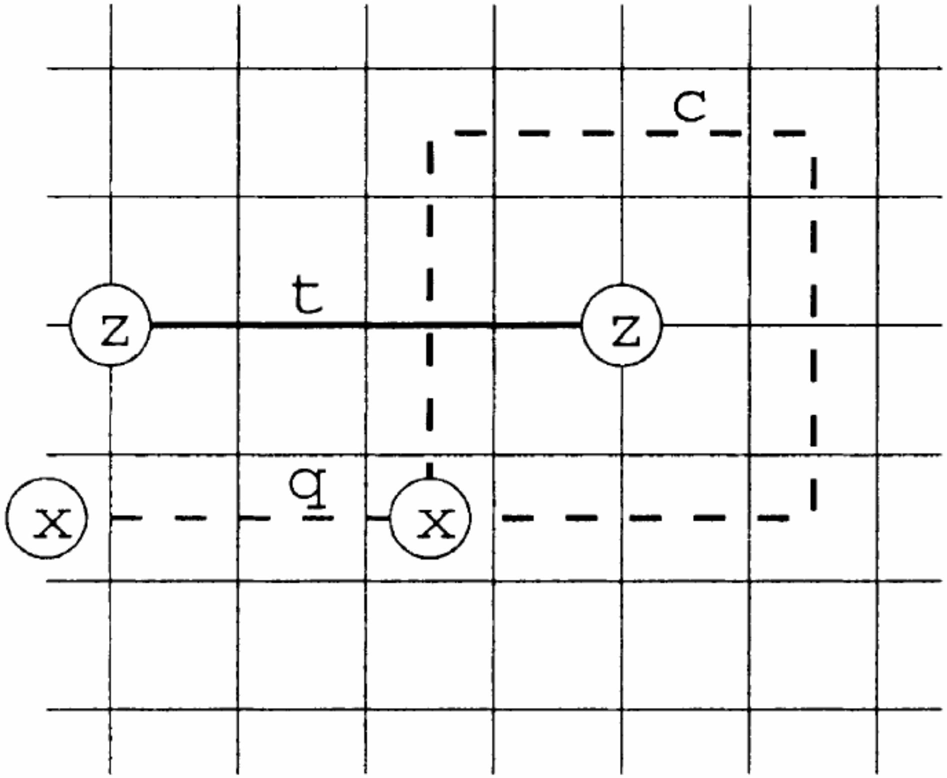
Phys. Rev. Lett. 90, 016803 (2003)

$$H_W = -K \sum_i \sigma_i^x \sigma_{i+\hat{e}_x}^y \sigma_{i+\hat{e}_x+\hat{e}_y}^x \sigma_{i+\hat{e}_y}^y$$



Properties of the toric code model



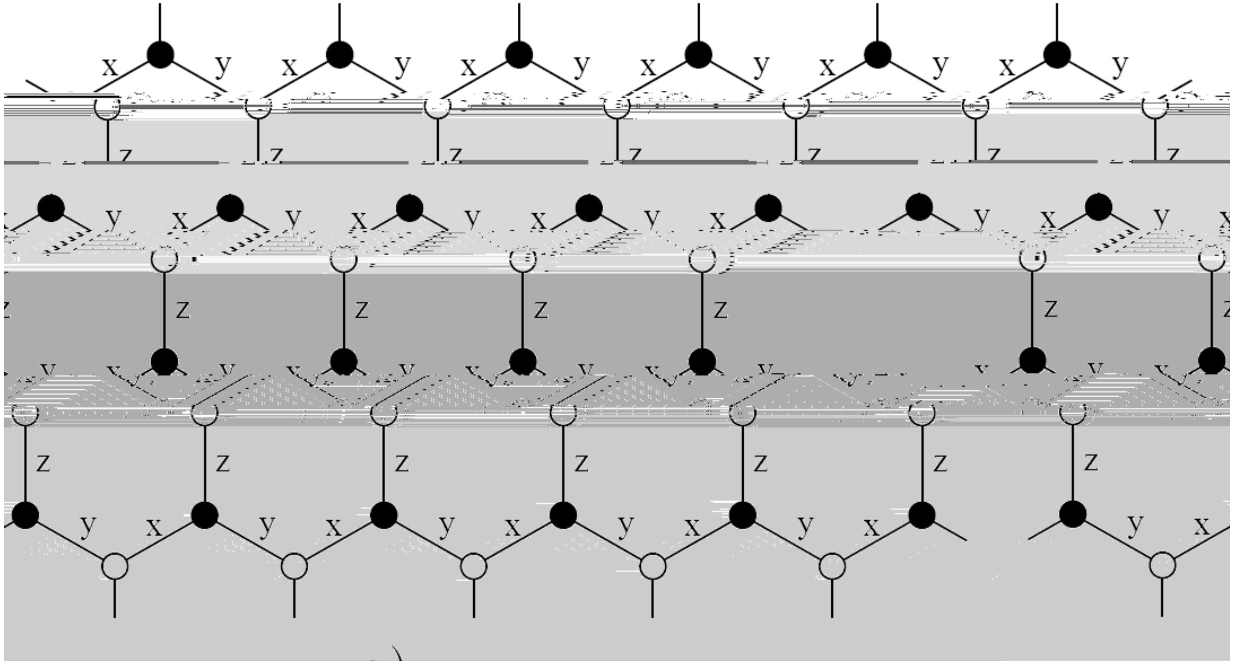


Two types of vortices of the low energy excitations.

The statistics of the vortex excitations – Abelian anyons

Kitaev spin-1/2 model

$$H = J_1 \sum_{x-link} \sigma_n^x \sigma_m^x + J_2 \sum_{y-link} \sigma_n^y \sigma_m^y + J_3 \sum_{z-link} \sigma_n^z \sigma_m^z$$

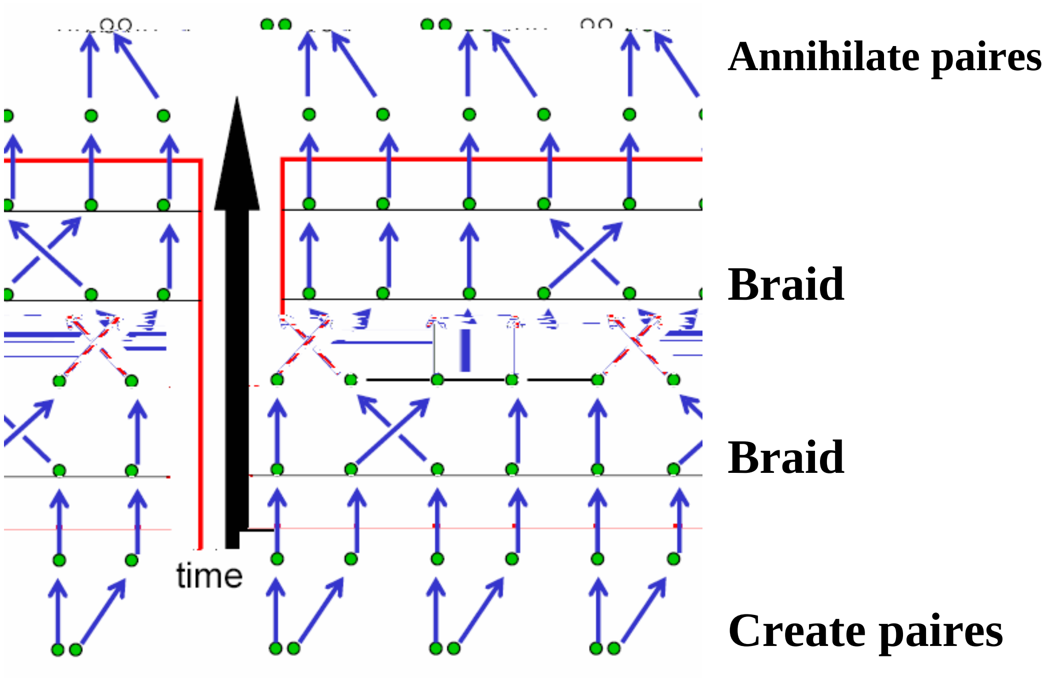


A. Kitaev, Ann Phys 321, 2 (2006).

Why interesting(I)
Topological Quantum Computation

- Error correction and fault-tolerance are essential in the operation of large scale quantum computers
- non-Abelian anyons: topological, resistant to local perturbation

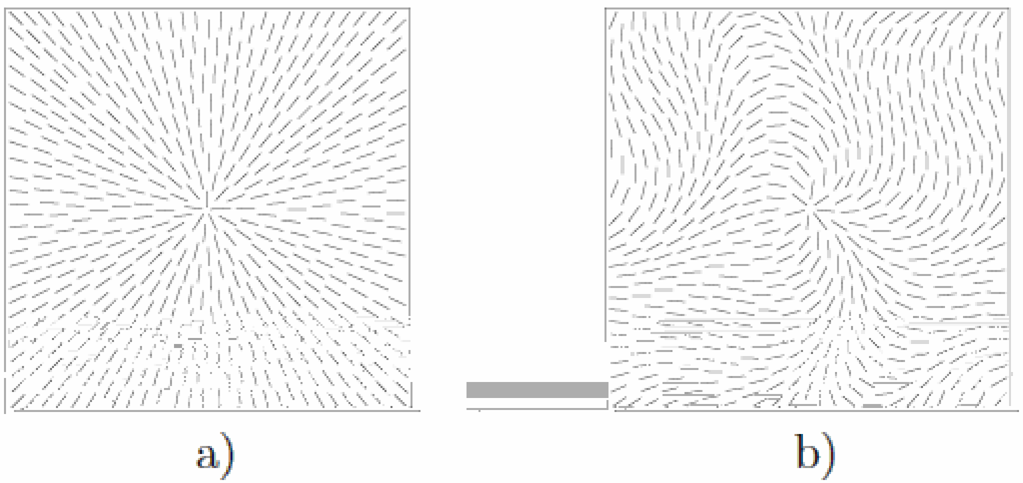
  fractional QHE 5/2, 12/5
- Microsoft Project Q



Why Interesting(II)

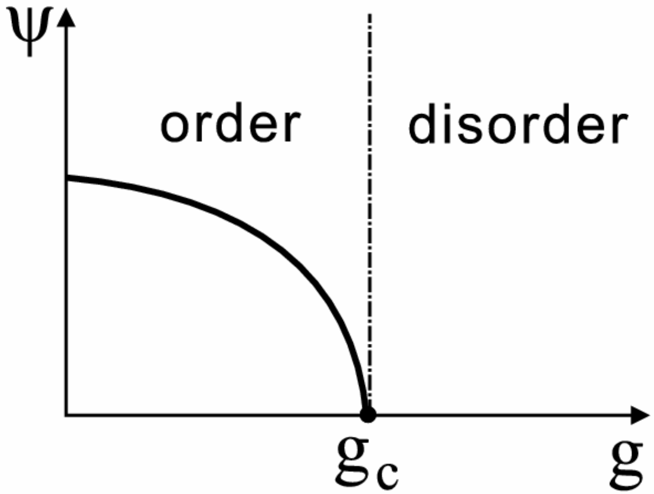
Quantum Phase Transitions and Topological Excitations

- Exact analytic solution in 2D in the ground state
- Ideal model for studying topological ordering and continuous quantum phase transitions
- Anyons as a kind of topological defects reveal nontrivial properties of the ground state.



A classical vortex (a) distorted by fluctuations (b)

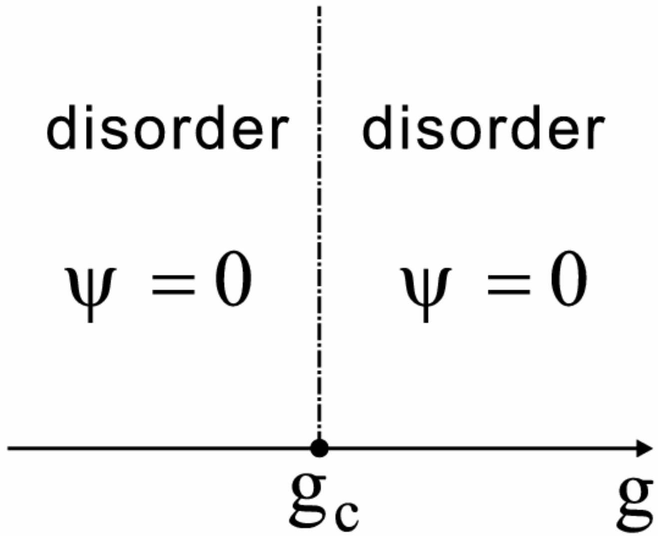
Continuous quantum phase transitions



Conventional: Landau-type

- Spontaneous symmetry breaking
- Local order parameters

Ferromagnet – Paramagnet



Topological:

- Both phases are gapped
- No symmetry breaking
- No local order parameters

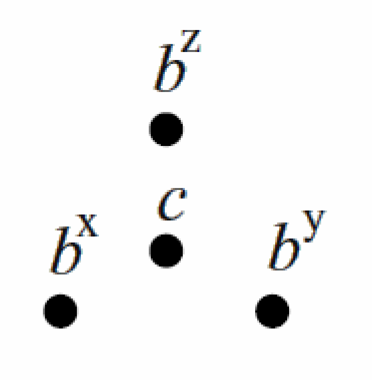
Fractional quantum Hall liquids

4 Majorana Fermion Representation of Pauli Matrices

$$\sigma_j^x = i b_j^x c_j$$

$$\sigma_j^y = i b_j^y c_j$$

$$\sigma_j^z = i b_j^z c_j$$



$$\{a_i, a_j\} = 2\delta_{ij}$$

$$a_i^2 = 1$$

Ettore Majorana

c_j, b_j^x, b_j^y, b_j^z are Majorana fermion operators

Physical s-1/2 spin: 2 degrees of freedom per spin

Each Majorana fermion has $2^{1/2}$ degree of freedom

4 Majorana fermions have totally 4 degrees of freedom

4 Majorana Fermion Representation of Kitaev Model

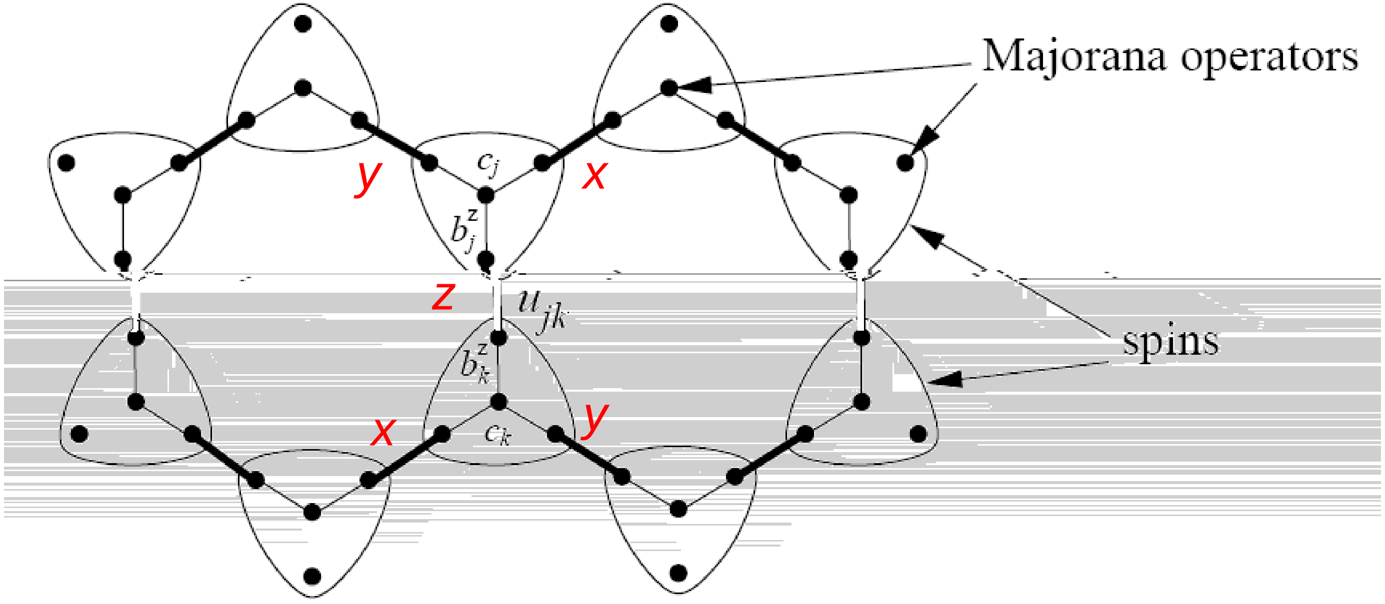
$$H = i\sum_{\langle jk \rangle} J_\alpha u_{jk}^\alpha c_j c_k \qquad \alpha = x, y, z$$

$$u_{jk}^\alpha = i b_j^\alpha b_k^\alpha \qquad \left(u_{jk}^\alpha\right)^2 = 1$$ ← Good quantum number

$$\sigma_j^x = i b_j^x c_j$$

$$\sigma_j^y = i b_j^y c_j$$

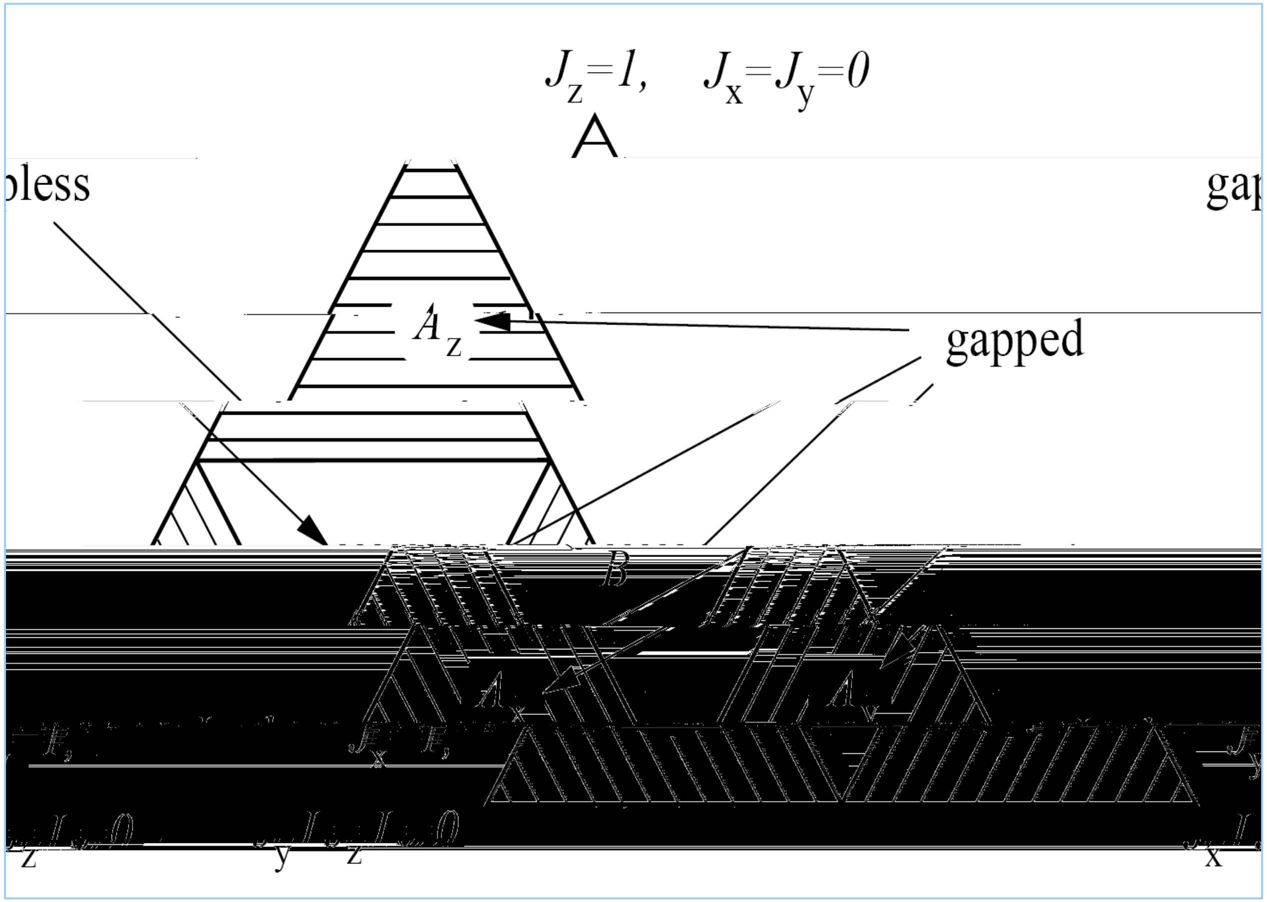
$$\sigma_j^z = i b_j^z c_j$$



2D Ground State Phase Diagram

The ground state is in a zero-flux phase (highly degenerate, $u_{jk} = 1$), the Hamiltonian can be rigorously diagonalized

- Non-Abelian anyons appear in the B phase in the presence of magnetic field.
- Abelian anyons exist in the gapped A phase.



4 Majorana Fermion Representation: constraint

$$\sigma_j^x \sigma_j^y = i\sigma_j^z$$

$$D_j = b_j^x b_j^y b_j^z c_j = 1$$

$$P = \prod_j \frac{1+D_j}{2}$$

$$|\psi_{phys}\rangle = P|\psi\rangle$$

Eigen-function in the extended Hilbert space

$$\sigma_j^x = i b_j^x c_j$$

$$\sigma_j^y = i b_j^y c_j$$

$$\sigma_j^z = i b_j^z c_j$$

3 Majorana Fermion Representation of Pauli Matrices

$$\boxed{D_j = b_j^x b_j^y b_j^z c_j = 1}$$

$$\sigma_j^x = i b_j^x c_j$$

$$\sigma_j^y = i b_j^y c_j$$

$$\sigma_j^z = i b_j^z c_j$$

$$\longrightarrow$$

$$\sigma_j^x = i b_j^x c_j$$

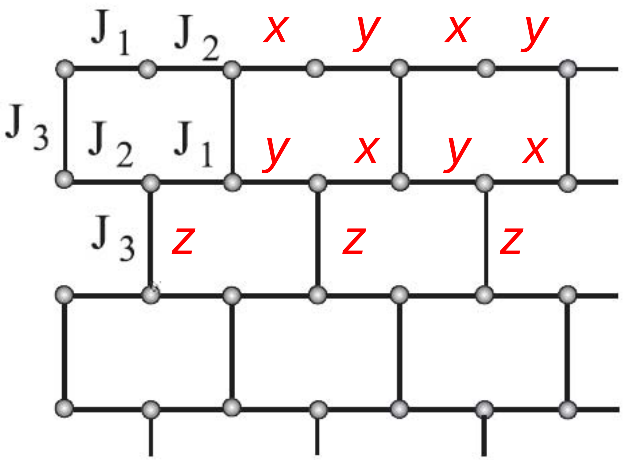
$$\sigma_j^y = i b_j^y c_j$$

$$\sigma_j^z = i b_j^y b_j^x$$

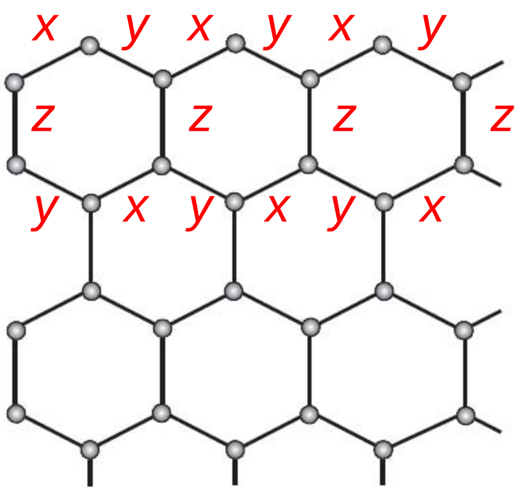
Totally $2^{3/2}$ degrees of freedom, still has a hidden $2^{1/2}$ redundant degree of freedom

Kitaev Model on a Brick-Wall Lattice

$$H = J_1 \sum_{x-link} \sigma_n^x \sigma_m^x + J_2 \sum_{y-link} \sigma_n^y \sigma_m^y + J_3 \sum_{z-link} \sigma_n^z \sigma_m^z$$



Brick-Wall Lattice



Honeycomb Lattice

$$H = \sum_{i+j=even} \left(J_1 \sigma_{i,j}^x \sigma_{i+1,j}^x + J_2 \sigma_{i-1,j}^y \sigma_{i,j}^y + J_3 \sigma_{i,j}^z \sigma_{i,j+1}^z \right)$$

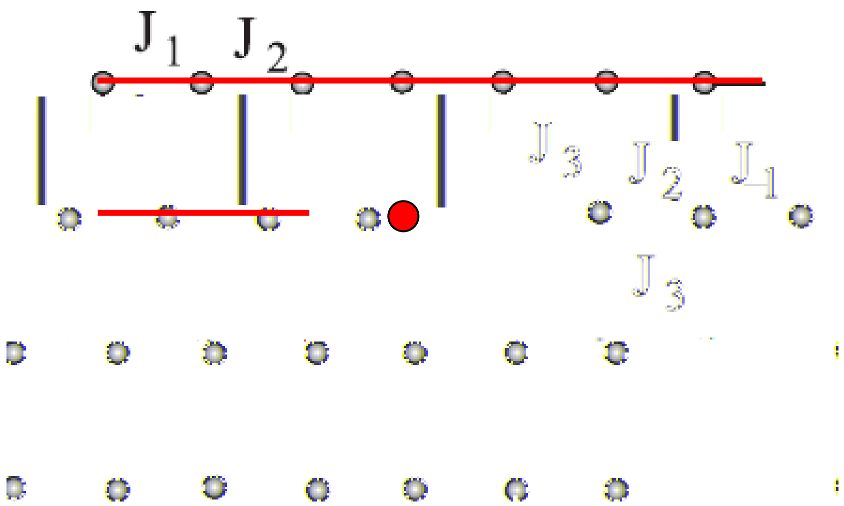
Jordan-Wigner Transformation

$$\sigma^{+}_{i,j} = 2a^{+}_{i,j} e^{i\pi\left(\sum_{k<j,l} a^{+}_{l,k} a_{l,k} + \sum_{l<i} a^{+}_{l,j} a_{l,j}\right)}$$

$$\sigma^{z}_{i,j} = 2a^{+}_{i,j} a_{i,j} - 1$$

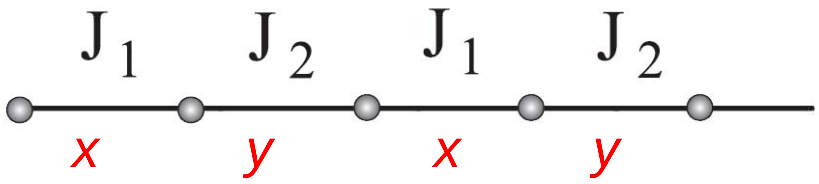
P. Jordan

E.P. Wigner



Represent spin operators by spinless fermion operators

Along Each Horizontal Chain



$$H=\sum_i\left(J_1\sigma^x_{2i-1}\sigma^x_{2i}+J_2\sigma^y_{2i}\sigma^y_{2i+1}\right)$$

$$=\sum_i J_1\left(a^+_{2i-1}-a_{2i-1}\right)\left(a^+_{2i}+a_{2i}\right)+J_2\left(a^+_{2i}+a_{2i}\right)\left(a^+_{2i+1}-a_{2i+1}\right)$$

Two Majorana Fermion Representation

$$c_i = i(a_i^+ - a_i), \quad d_i = a_i^+ + a_i \qquad i = odd$$

$$d_i = i(a_i^+ - a_i), \quad c_i = a_i^+ + a_i \qquad i = even$$

$$\begin{aligned} H &= \sum_i \left(J_1 \sigma_{2i-1}^x \sigma_{2i}^x + J_2 \sigma_{2i}^y \sigma_{2i+1}^y \right) \\ &= -i \sum_i \left(J_1 c_{2i-1} c_{2i} - J_2 c_{2i} c_{2i+1} \right) \end{aligned}$$

Onle c_i-type Majorana fermion operators appear!

Two Majorana Fermion Representation

$$c_{ij} = i(a_{ij}^{+} - a_{ij}) \quad d_{ij} = a_{ij}^{+} + a_{ij} \quad i + j = odd$$

$$d_{ij} = i(a_{ij}^{+} - a_{ij}) \quad c_{ij} = a_{ij}^{+} + a_{ij} \quad i + j = even$$

c_i and d_i are Majorana fermion operators

A conjugate pair of fermion operators is represented by two Majorana fermion operators

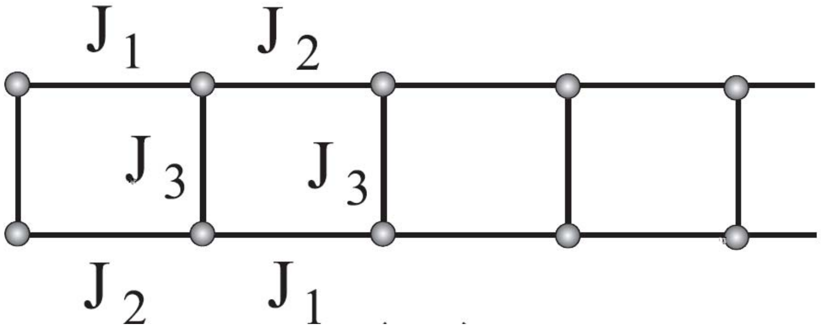
No redundant degrees of freedom!

Vertical Bond

$$\sigma_i^z = ic_i d_i$$

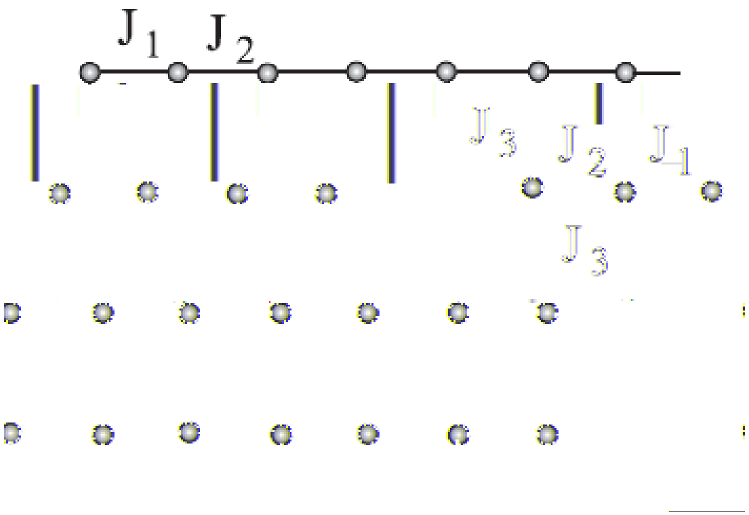
$$\sigma_i^z \sigma_j^z = \left(ic_i d_i\right)\left(ic_j d_j\right)$$

No Phase String



2 Majorana Representation of Kitaev Model

$$H = \sum_{i+j=even} \left(J_1 \sigma^x_{i,j} \sigma^x_{i+1,j} + J_2 \sigma^y_{i-1,j} \sigma^y_{i,j} + J_3 \sigma^z_{i,j} \sigma^z_{i,j+1} \right)$$
$$= -i \sum_{i+j=even} \left(J_1 c_{i,j} c_{i+1,j} - J_2 c_{i-1,j} c_{i,j} + J_3 D_{i,j} c_{i,j} c_{i,j+1} \right)$$

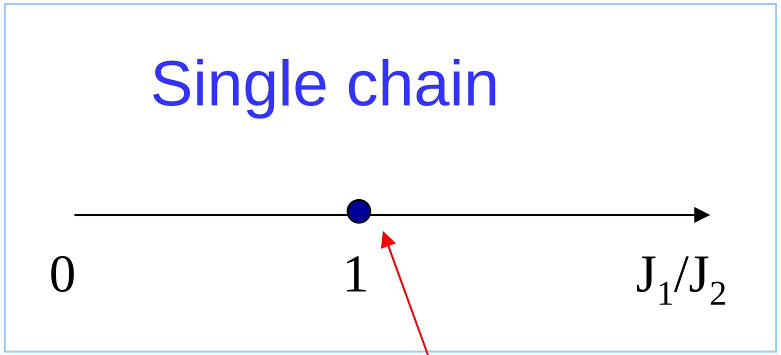


$$D_{i,j} = i d_{i,j} d_{i,j+1}$$

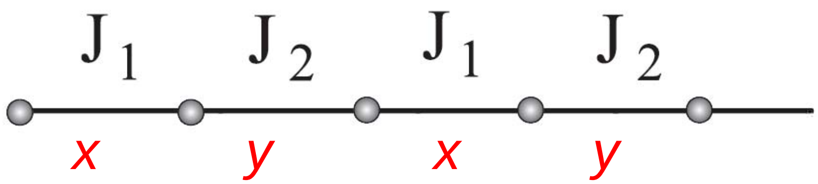
good quantum numbers

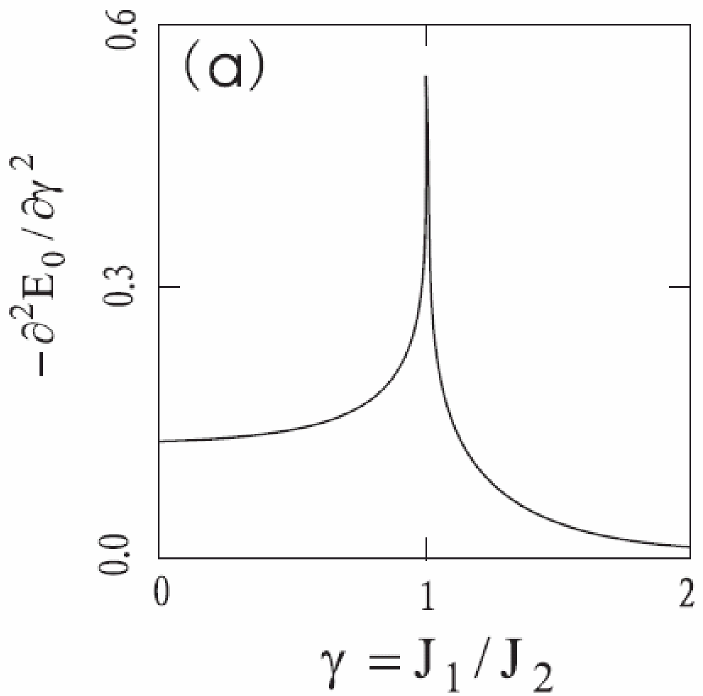
Ground state is in a zero-flux phase $D_{i,j} = D_{0,j}$

Phase Diagram



Critical point



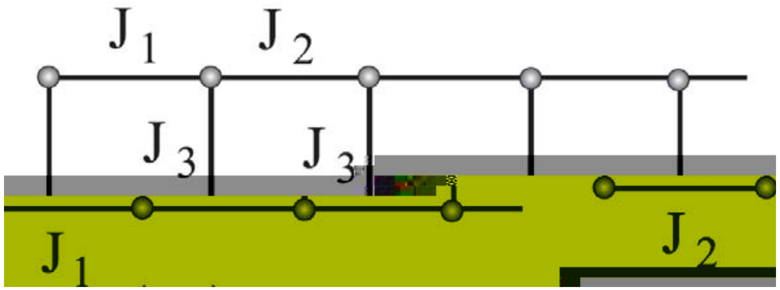


Quasiparticle excitation:

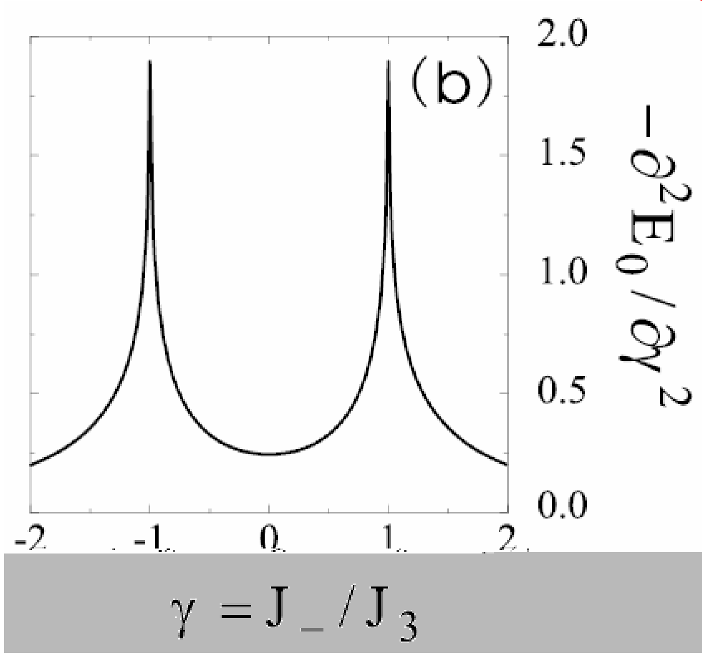
$$\varepsilon_{k,\pm} = \pm\sqrt{J_1^2 + J_2^2 + 2J_1 J_2 \cos k}$$

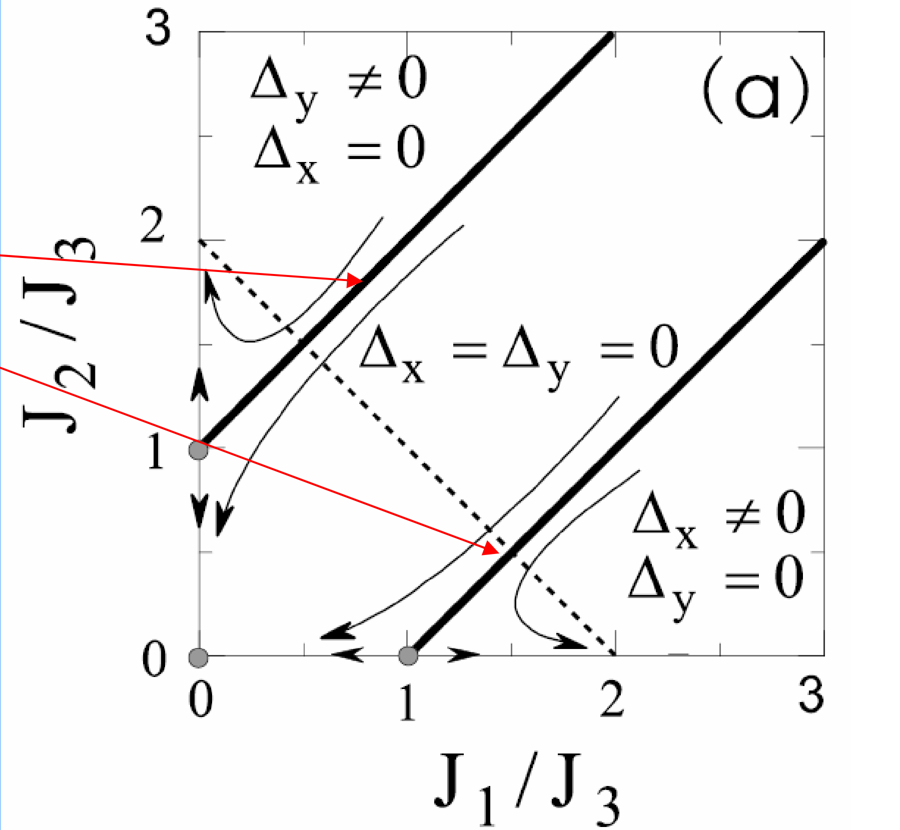
Ground state energy $E_0 = \sum_k \varepsilon_{k,-}$

Phase Diagram



Critical lines





Two-leg ladder

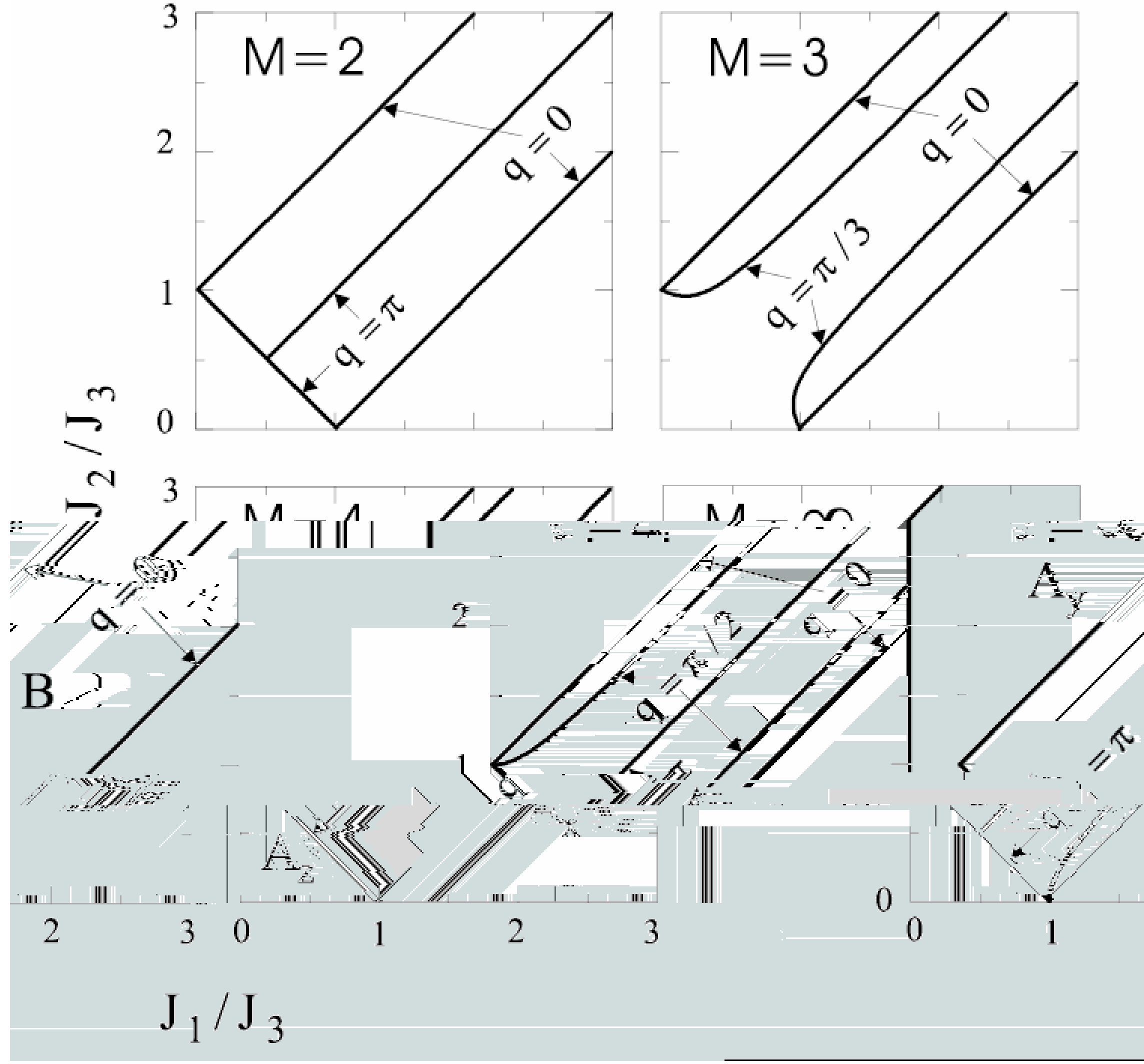
Multi-Chain System

Chain number = 2 M

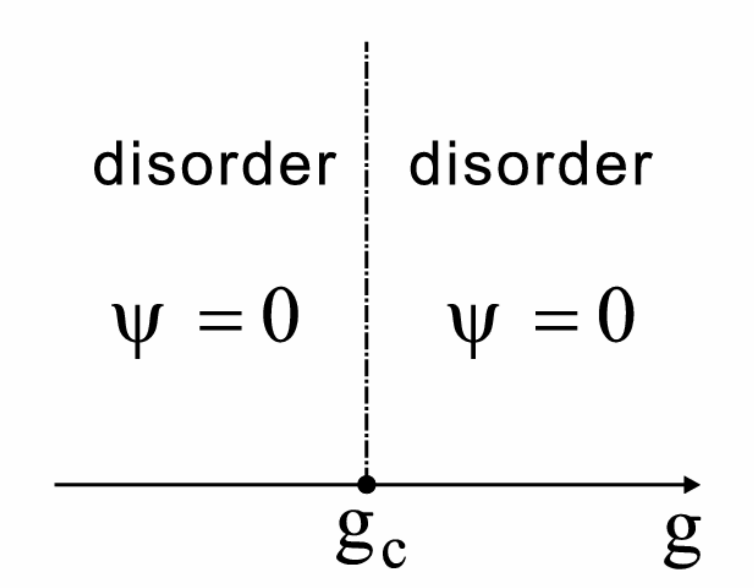
Thick solid lines as critical lines

Infinite critical lines form quantum critical regime

How to characterize these quantum phase transitions?



Classifications of quantum phase transitions



Topological:

- Both phases are gapped
- No symmetry breaking
- No local order parameters

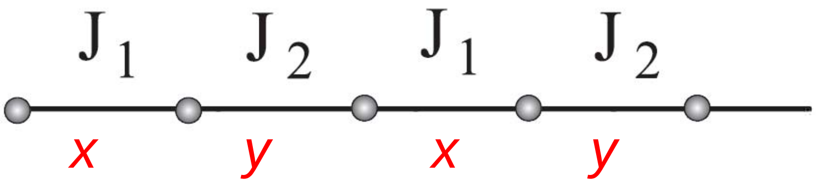
Questions:

- Can one find certain type of nonlocal order parameters to describe such topological phase transition?

It is true in the abelian gapped phase. Duality is important!

QPT: Single Chain

$$H = \sum_i \left(J_1 \sigma_{2i-1}^x \sigma_{2i}^x + J_2 \sigma_{2i}^y \sigma_{2i+1}^y \right)$$



Duality Transformation

$$\sigma_j^x = \tau_{j-1}^x \tau_j^x \quad , \quad \sigma_j^y = \prod_{k=j}^{2N} \tau_k^y$$

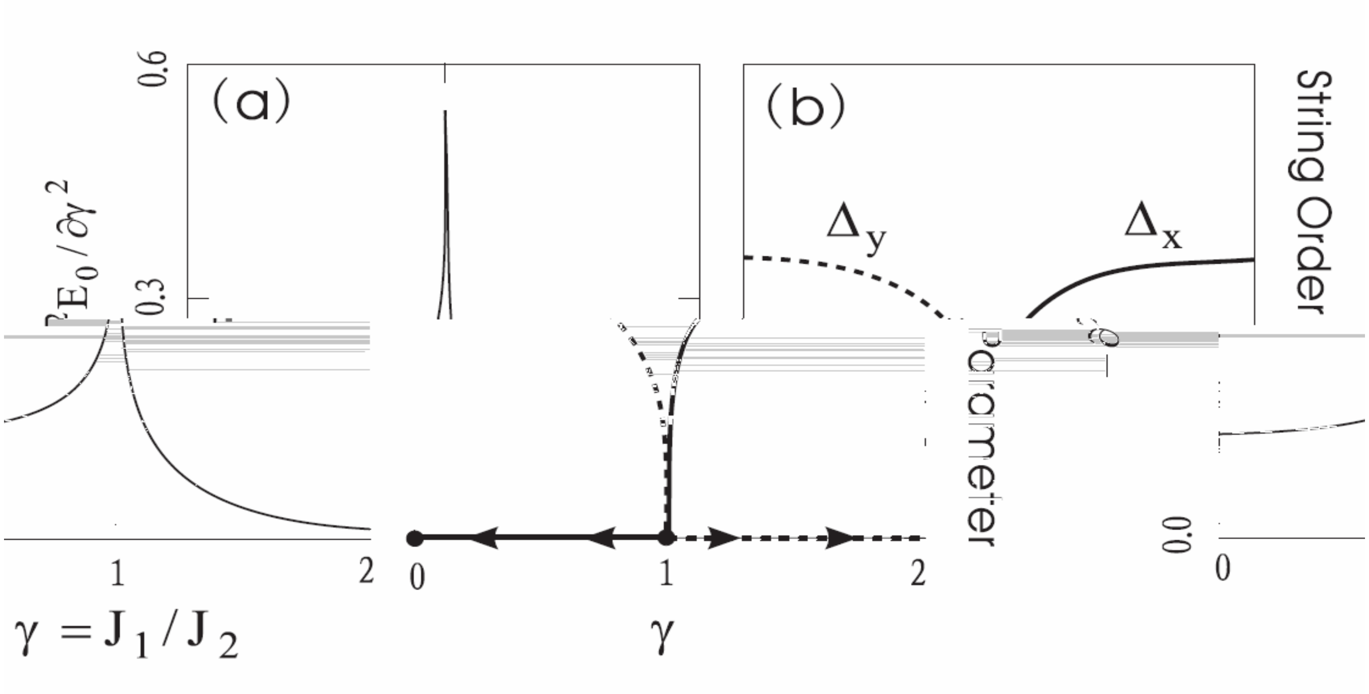
$$\tau_j^y = \sigma_j^y \sigma_{j+1}^y \quad , \quad \tau_j^x = \prod_{k=1}^{j} \sigma_k^x$$

$$H = \sum_i \left(J_1 \tau_{2i-2}^x \tau_{2i}^x + J_2 \tau_{2i}^y \right)$$

This is a self-dual model.

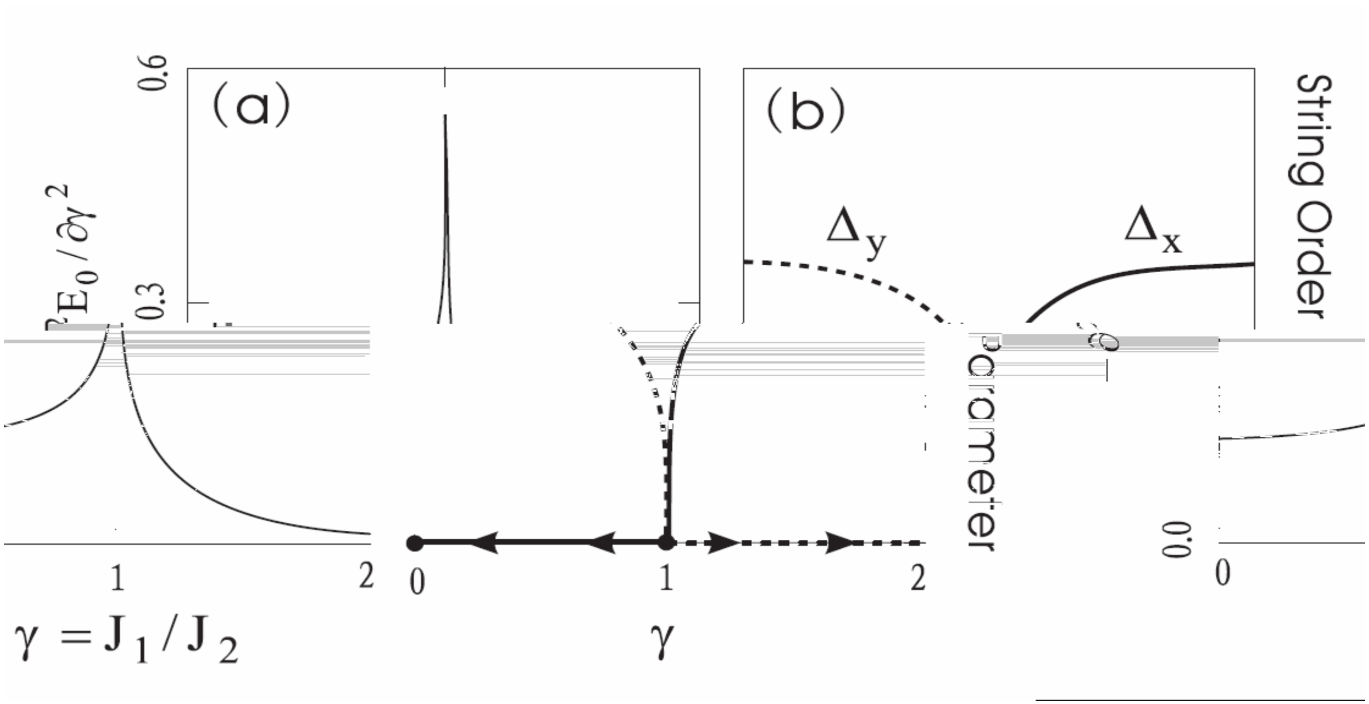
Non-local String Order Parameter

$$\Delta_x \equiv \lim_{n\to\infty}\left\langle \sigma_1^x\sigma_2^x\cdots\sigma_{2n}^x\right\rangle \sim \lim_{n\to\infty}\left\langle \tau_0^x\tau_{2n}^x\right\rangle \sim \begin{cases} \left[1-\left(J_2/J_1\right)^2\right]^{1/4} & , \quad J_1 > J_2 \\ 0 & , \quad J_1 \le J_2 \end{cases}$$

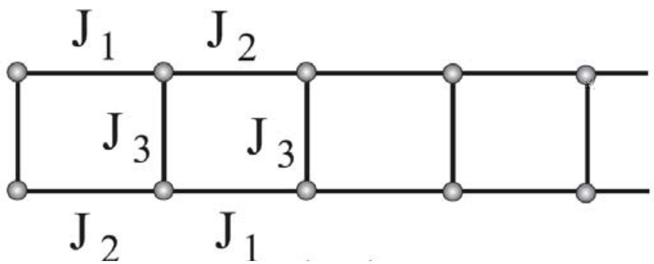


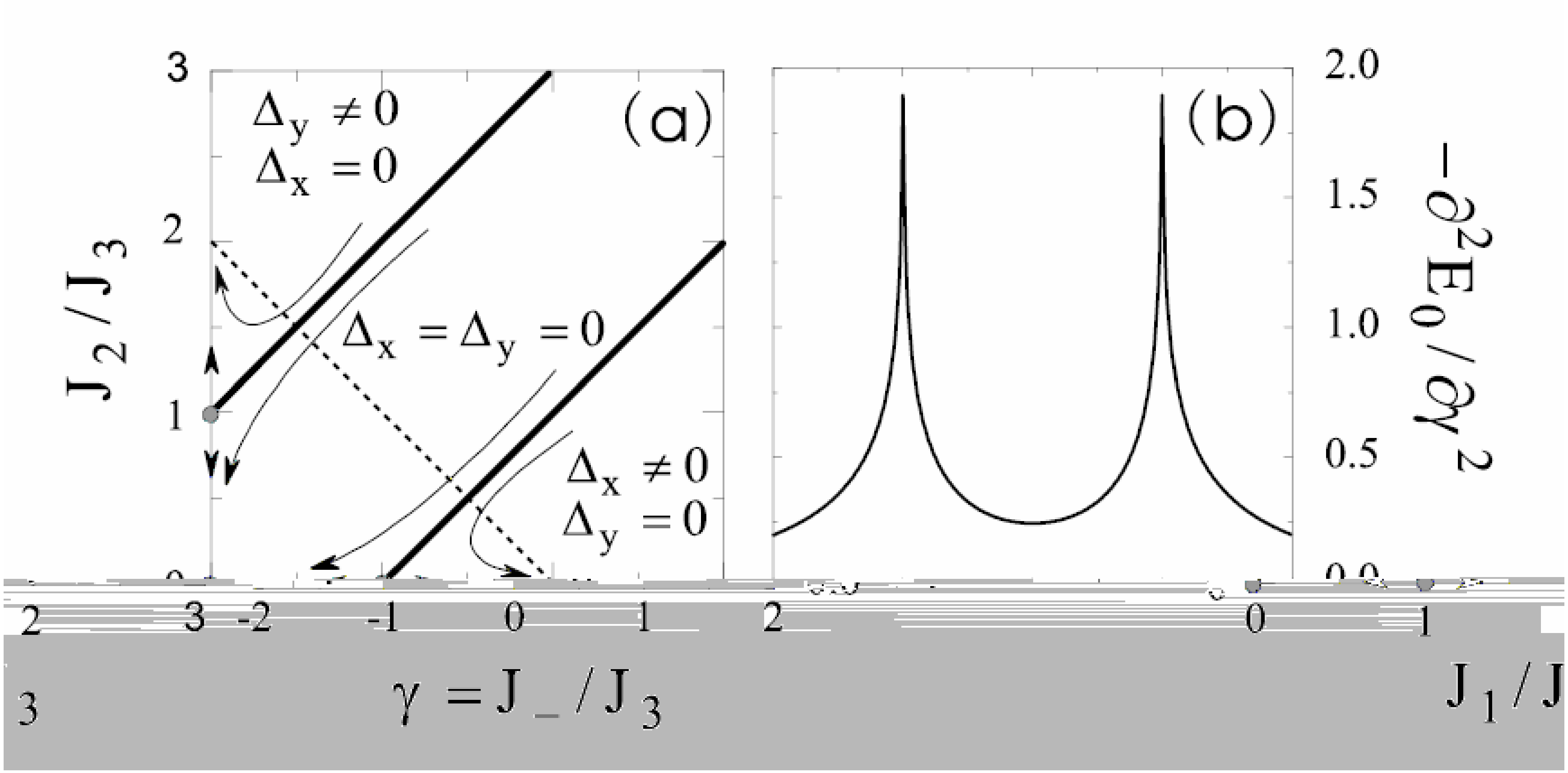
Another String Order Parameter

$$\Delta_y \equiv \lim_{n\to\infty} \left\langle \sigma_2^y \sigma_3^y \cdots \sigma_{2n+1}^y \right\rangle \sim \begin{cases} \left[1-\left(J_1/J_2\right)^2\right]^{1/4} & , \quad J_1 < J_2 \\ 0 & , \quad J_1 \geq J_2 \end{cases}$$

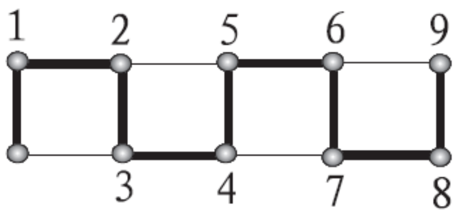


Two-leg ladder

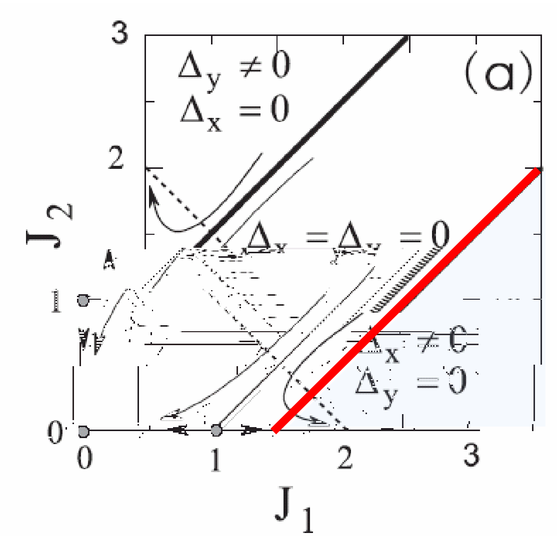




Phase I: $J_1 > J_2 + J_3$



$$H = \sum_i \left(J_1 \sigma^x_{2i-1} \sigma^x_{2i} + J_2 \sigma^y_{2i} \sigma^y_{2i+3} + J_3 \sigma^z_{2i} \sigma^z_{2i+1} \right)$$



In the dual space:

$$H = \sum_i \left(J_1 \tau^x_{2i-2} \tau^x_{2i} + J_2 W_i \tau^y_{2i-2} \tau^y_{2i} + J_3 \tau^z_{2i} \right)$$

$$W_i = \tau^x_{2i-3} \tau^z_{2i-1} \tau^x_{2i+1}$$

$$\sigma^x_j = \tau^x_{j-1} \tau^x_j \quad , \quad \sigma^z_j = \prod_{k=j}^{2N} \tau^z_k$$

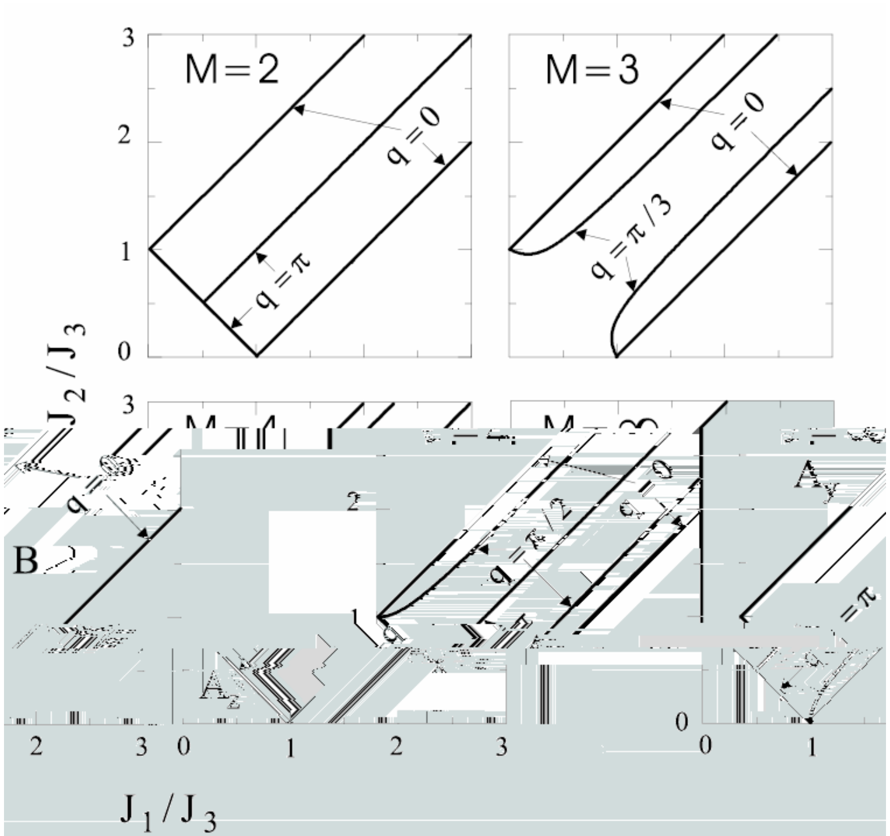
$$\tau^z_j = \sigma^z_j \sigma^z_{j+1} \quad , \quad \tau^x_j = \prod_{k=1}^{j} \sigma^x_k$$

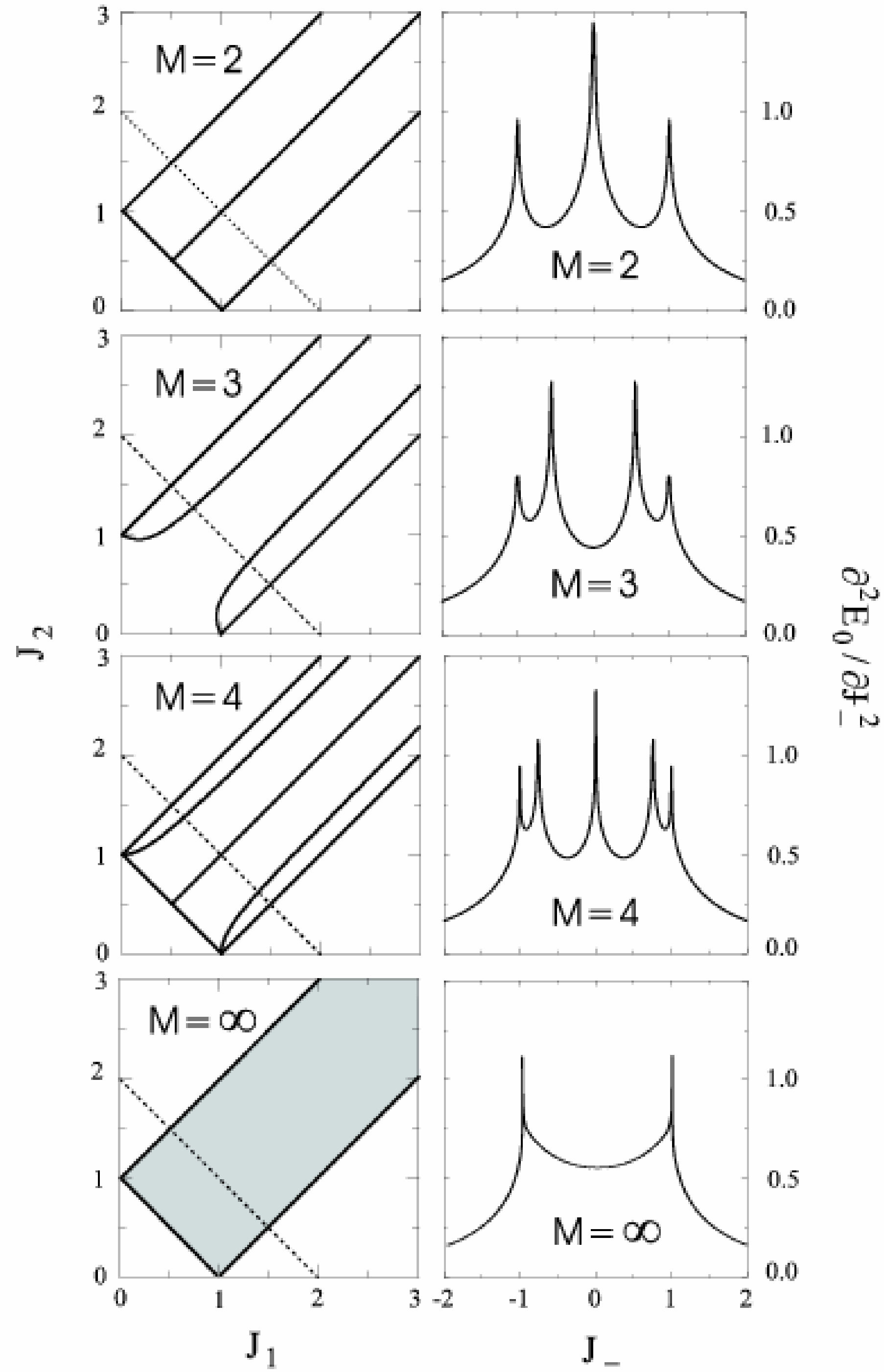
W_1 = -1 in the ground state

String Order Parameters

QPT: multi chains

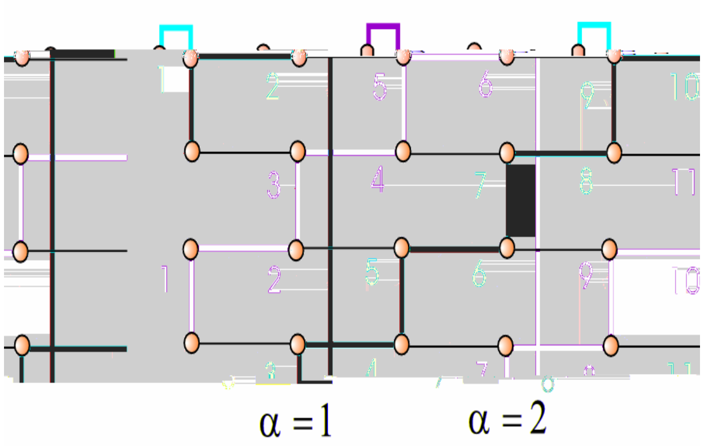
Chain number = 2 M

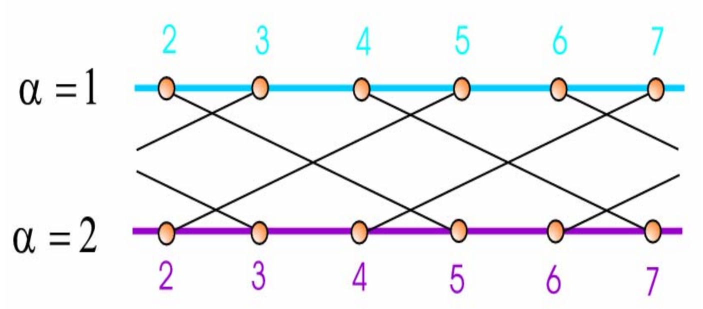




QPT in a multi-chain system

4-chain ladder $M = 2$





$$H = -i\sum_{i=1}^{2N}\sum_{\alpha=1}^{M}\left(J_1 c_{2i-1,\alpha}c_{2i,\alpha} - J_2 c_{2i,\alpha}c_{2i+3,\alpha+1} + J_3(-1)^i c_{2i,\alpha}c_{2i+1,\alpha}\right)$$

Fourier Transformation

$$H = -i \sum_{n=1}^{2N} \sum_{\alpha=1}^{M} \left(J_1 c_{2n-1,\alpha} c_{2n,\alpha} - J_2 c_{2n,\alpha} c_{2n+3,\alpha+1} + J_3 (-1)^n c_{2n,\alpha} c_{2n+1,\alpha} \right)$$

$$c_{i,\alpha} = \frac{1}{\sqrt{M}} \sum_q e^{iqr_i} c_{i,q}$$

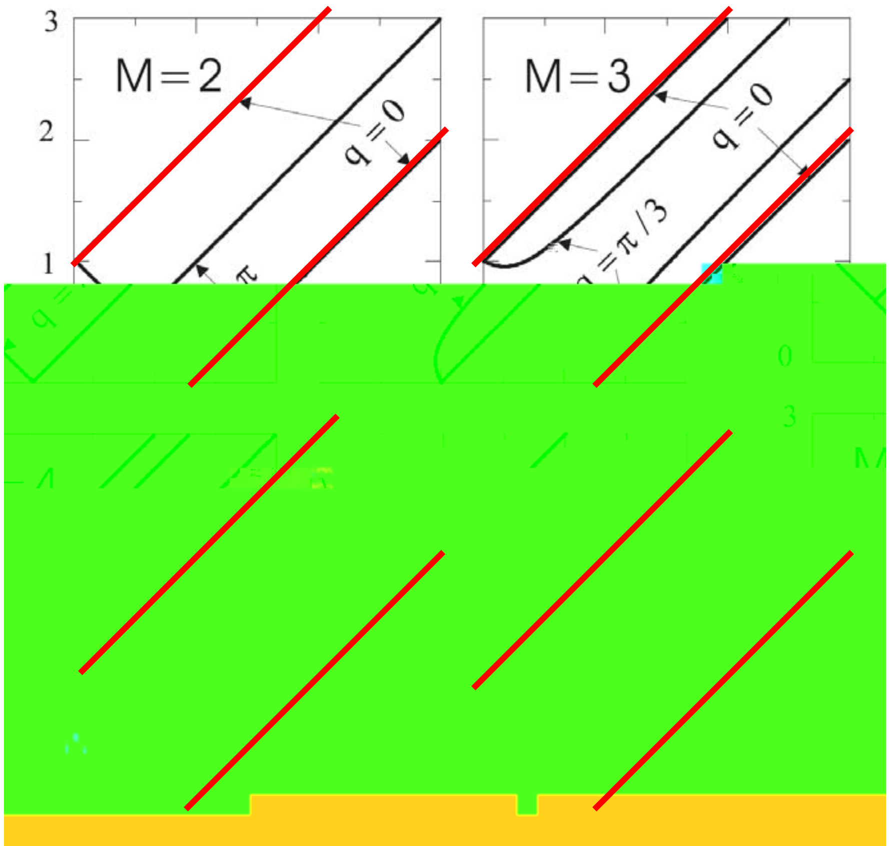
$$q = \frac{2\pi m}{M}, \qquad m = 0,1,\ldots,M-1$$

$$H = \sum_q H_q$$

$$H_q = -i \sum_i \left(J_1 c_{2i-1,-q} c_{2i,q} - J_2 e^{iq} c_{2i,-q} c_{2i+3,q} + J_3 (-1)^i c_{2i,-q} c_{2i+1,q} \right)$$

$q = 0$

$$H_{q=0} = -i\sum_i \left(J_1 c_{2i-1,0} c_{2i,0} - J_2 c_{2i,0} c_{2i+3,0} + J_3 (-1)^i c_{2i,0} c_{2i+1,0} \right)$$



$c_{i,0}$ is still a Majorana fermion operator

$H_{q=0}$ is exactly same as the Hamiltonian of a two-leg ladder

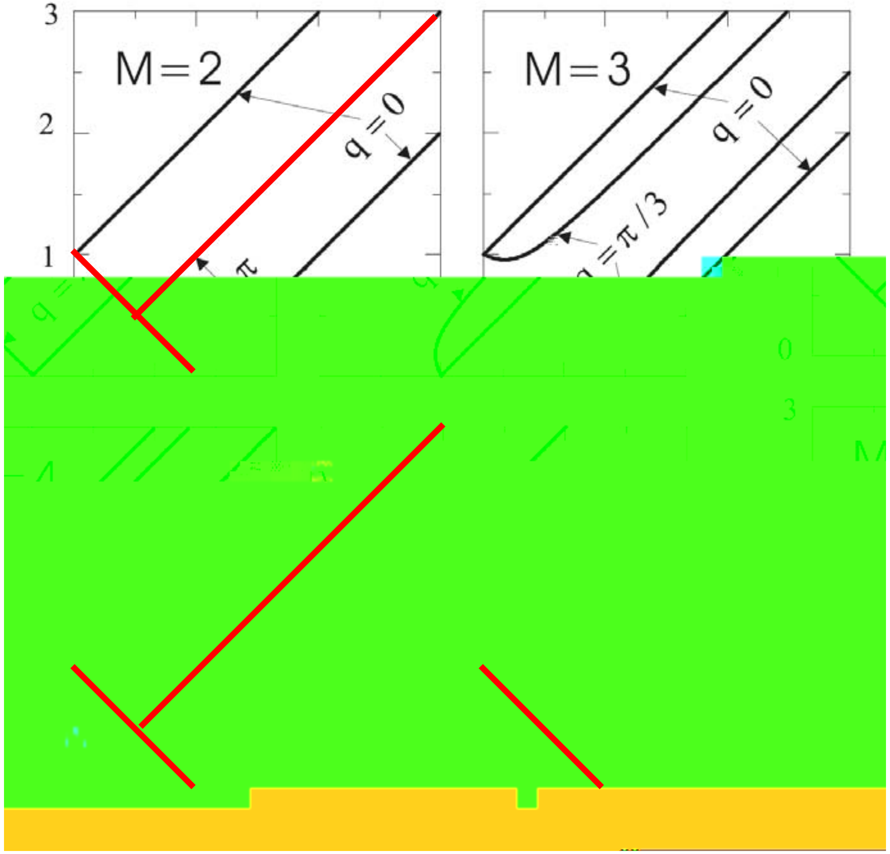
String Order Parameter

$$\Delta_{x,0} = \lim_{i\to\infty}(-1)^i \left\langle c_{1,0}c_{2,0}\cdots c_{2i,0}\right\rangle \begin{cases} \neq 0 & J_- > J_3 \\ = 0 & J_- \leq J_3 \end{cases}$$

$$\Delta_{y,0} = \lim_{i\to\infty}(-1)^i \left\langle c_{2,0}c_{3,0}\cdots c_{2i+1,0}\right\rangle \begin{cases} \neq 0 & J_- < -J_3 \\ = 0 & J_- \geq -J_3 \end{cases}$$

$q = \pi$

$$H_{q=\pi} = -i\sum_i \left(J_1 c_{2i-1,\pi} c_{2i,\pi} + J_2 c_{2i,\pi} c_{2i+3,\pi} + J_3 (-1)^i c_{2i,\pi} c_{2i+1,\pi} \right)$$



$c_{i,\pi}$ is also a Majorana fermion operator

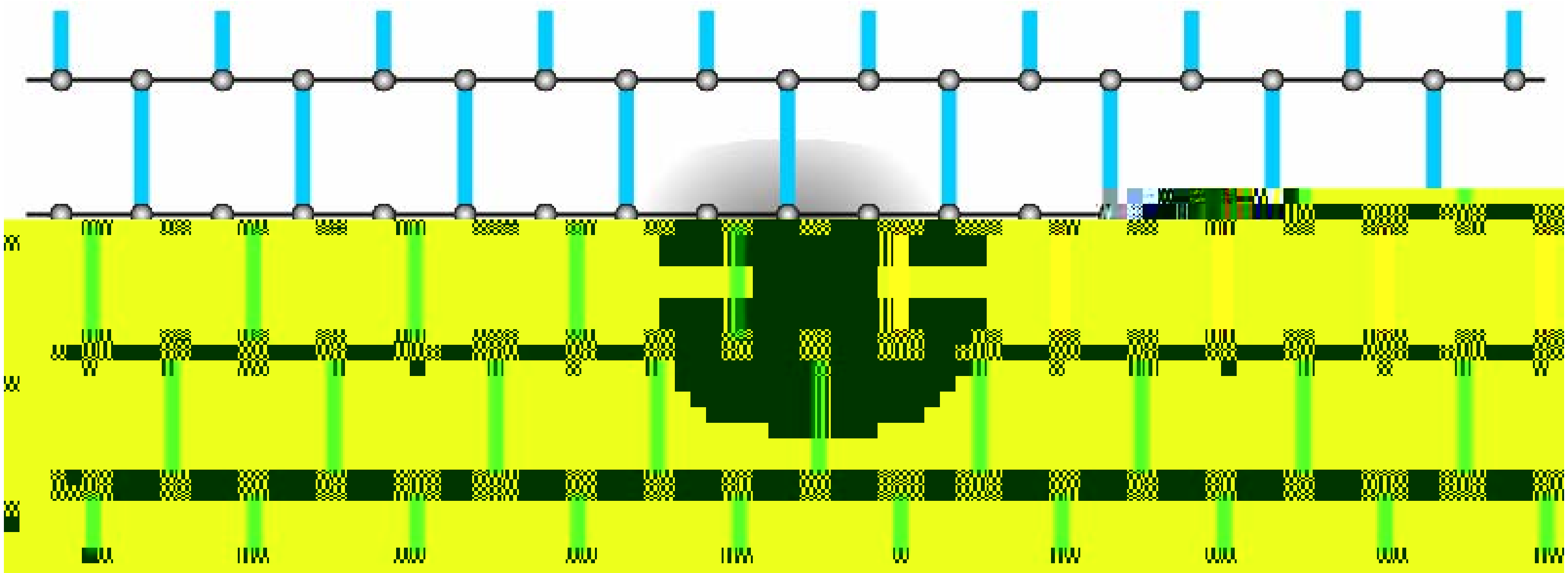
$H_{q=\pi}$ is also the same as the Hamiltonian of a two-leg ladder, only J_2 changes sign

String Order Parameter

$$\Delta_{x,\pi} = \lim_{n\to\infty} (-1)^n \left\langle c_{1,\pi} c_{2,\pi} \cdots c_{2n,\pi} \right\rangle \begin{cases} \neq 0 & J_+ > J_3, \quad J_1 > J_2 \\ = 0 & else \end{cases}$$

$$\Delta_{y,\pi} = \lim_{n\to\infty} (-1)^n \left\langle c_{2,\pi} c_{3,\pi} \cdots c_{2n+1,\pi} \right\rangle \begin{cases} \neq 0 & J_+ > J_3, \quad J_1 < J_2 \\ = 0 & else \end{cases}$$

Topological excitations in the Kitaev model



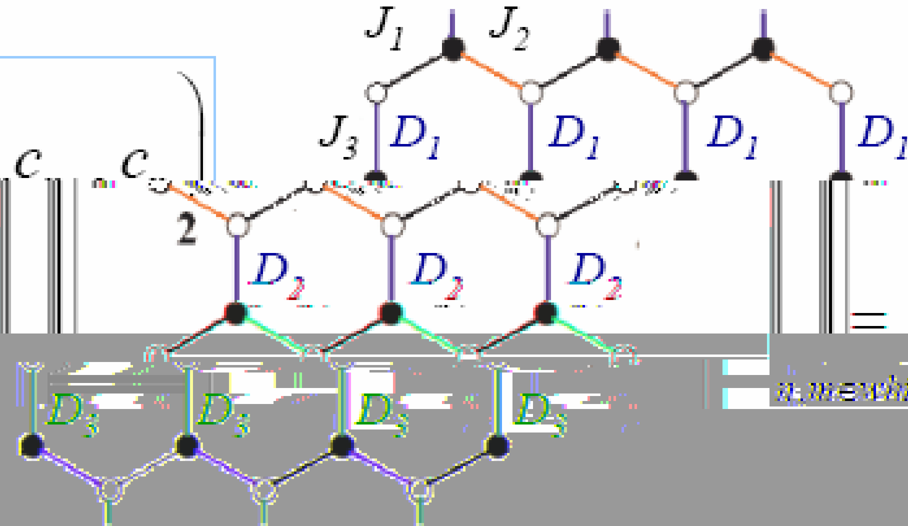
• Such vortex excitations in gapped A phases are the low-energy excited states and behave as Abelian anyonic excitations.

• In the external magnetic field, each vortex in the B phase behave as a non-Abelian anyon and carries a unpaired Majorana zero mode.

2D Ground State

$$H = -i \sum_{n \in white} \left(\sum_{\mu=1,2,3} J_\mu c_{n+e_\mu} c_n + J_3 D_n c_{n+e_3} c_n \right) = \sum_{n,m \in white} \Psi_n^+ H_{nm} \Psi_m$$

$$\Psi_n = \begin{pmatrix} c_n \\ c_{n+e_3} \end{pmatrix}$$



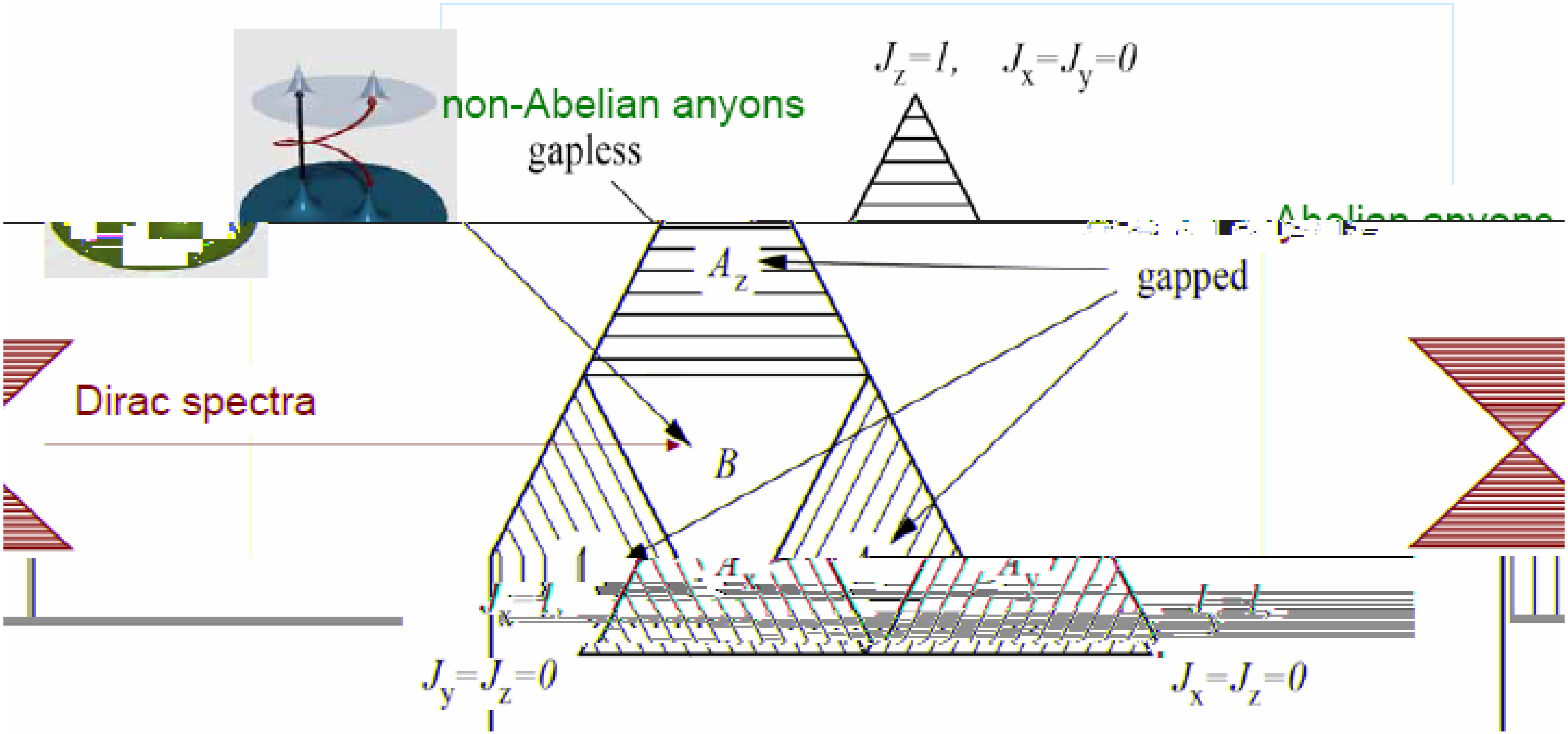
The Bloch matrix for $D_n = 1$ and $J_1 = J_2$

$$H(k) = h_1(k)\sigma_1 + h_2(k)\sigma_2$$

$$h_1(k) = J_1 \sin\frac{\sqrt{3}k_x + 3k_y}{2} - J_2 \sin\frac{\sqrt{3}k_x - 3k_y}{2}$$

$$h_2(k) = J_1 \cos\frac{\sqrt{3}k_x + 3k_y}{2} + J_2 \cos\frac{\sqrt{3}k_x - 3k_y}{2} + J_3$$

2D Ground State Phase Diagram



Two kinds of excitations

- Fermionic excitations
- Topological excitations: vortices (anyons)

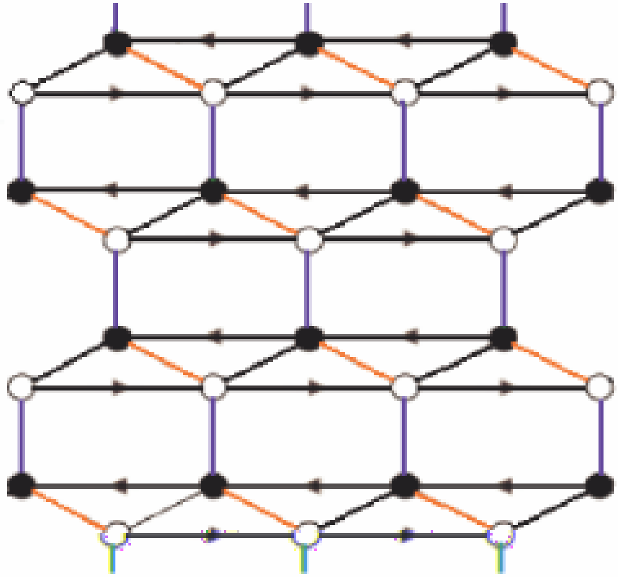
Gap the Fermionic excitations

$$H = -i \sum_{n \in white} \left(\sum_{\mu=1,2,3} J_\mu c_{n+e_\mu} c_n + J_3 D_n c_{n+e_3} c_n \right)$$

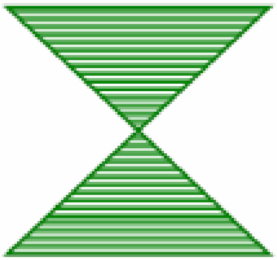
$$H_{tot} = H + H_{3-site}$$

$$H_{3-site} = J_4 \sum_{(ijk)\in\Delta} \sigma_i^y \sigma_j^z \sigma_k^x + J_4 \sum_{(ijk)\in\nabla} \sigma_i^x \sigma_j^z \sigma_k^y$$
$$= -iJ_4 \sum_{i \in white} c_i c_k + iJ_4 \sum_{i \in black} c_i c_k$$

Break time reversal symmetry

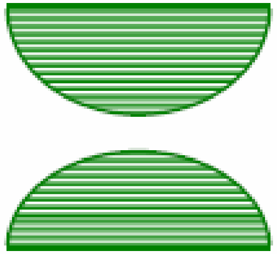


Gapless



H

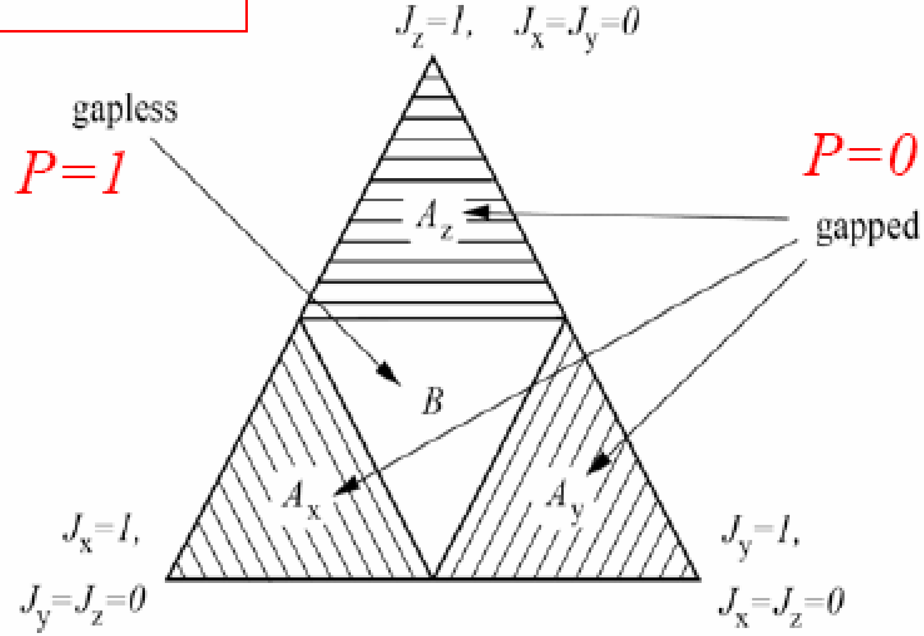
Gapped



H_{tot}

Topological index

$$P = \frac{1}{8\pi}\int d^2k\,\varepsilon^{\mu\nu}\hat{h}\cdot\left(\partial_{k_\mu}\hat{h}\times\partial_{k_\nu}\hat{h}\right)$$



A and B phases are topologically distinct

Bloch matrix:

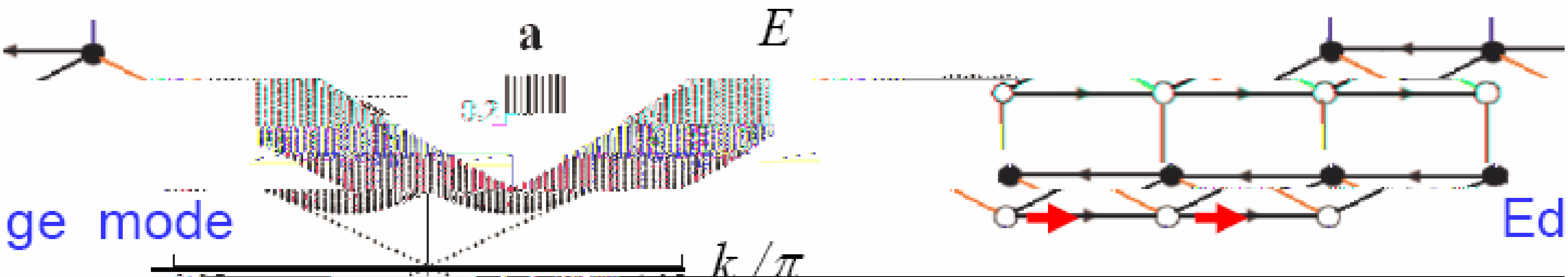
$$h_1(k) = J_1 \sin\frac{\sqrt{3}k_x + 3k_y}{2} - J_2 \sin\frac{\sqrt{3}k_x - 3k_y}{2}$$

$$h_2(k) = J_1 \cos\frac{\sqrt{3}k_x + 3k_y}{2} + J_2 \cos\frac{\sqrt{3}k_x - 3k_y}{2} + J_3$$

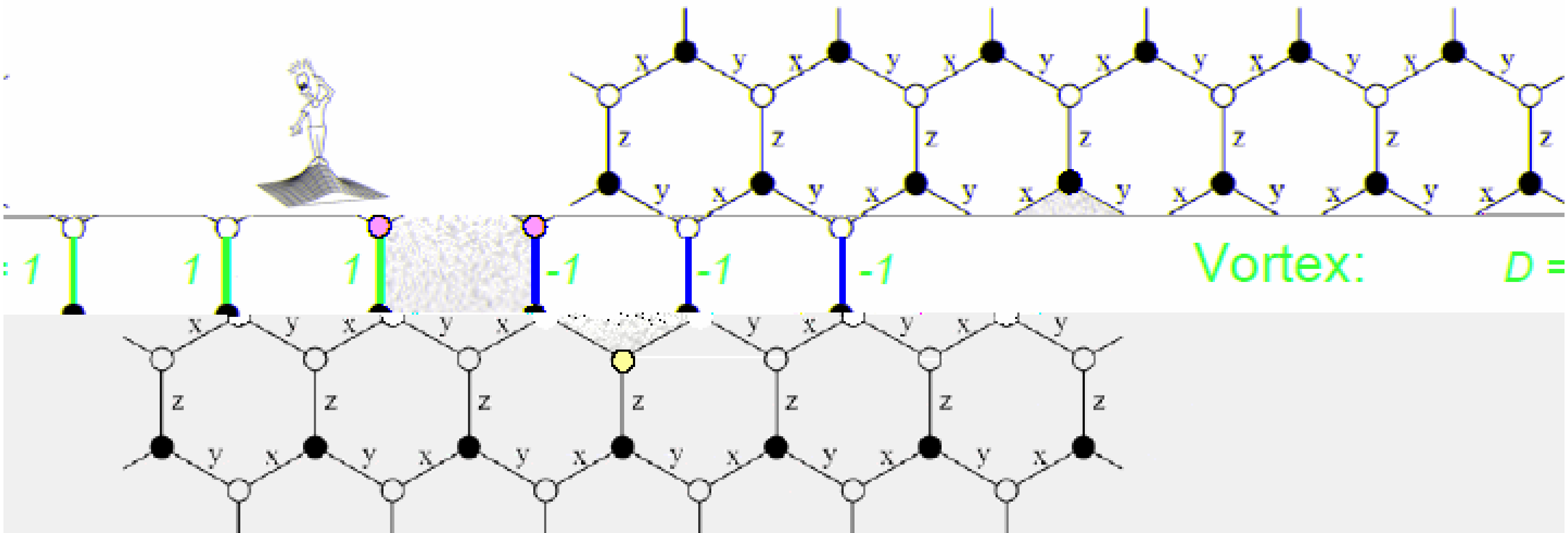
$$h_3(k) = 2J_4 \sin\sqrt{3}k_x$$

$$H_{tot}(k) = h_1(k)\sigma_1 + h_2(k)\sigma_2 + h_3(k)\sigma_3$$

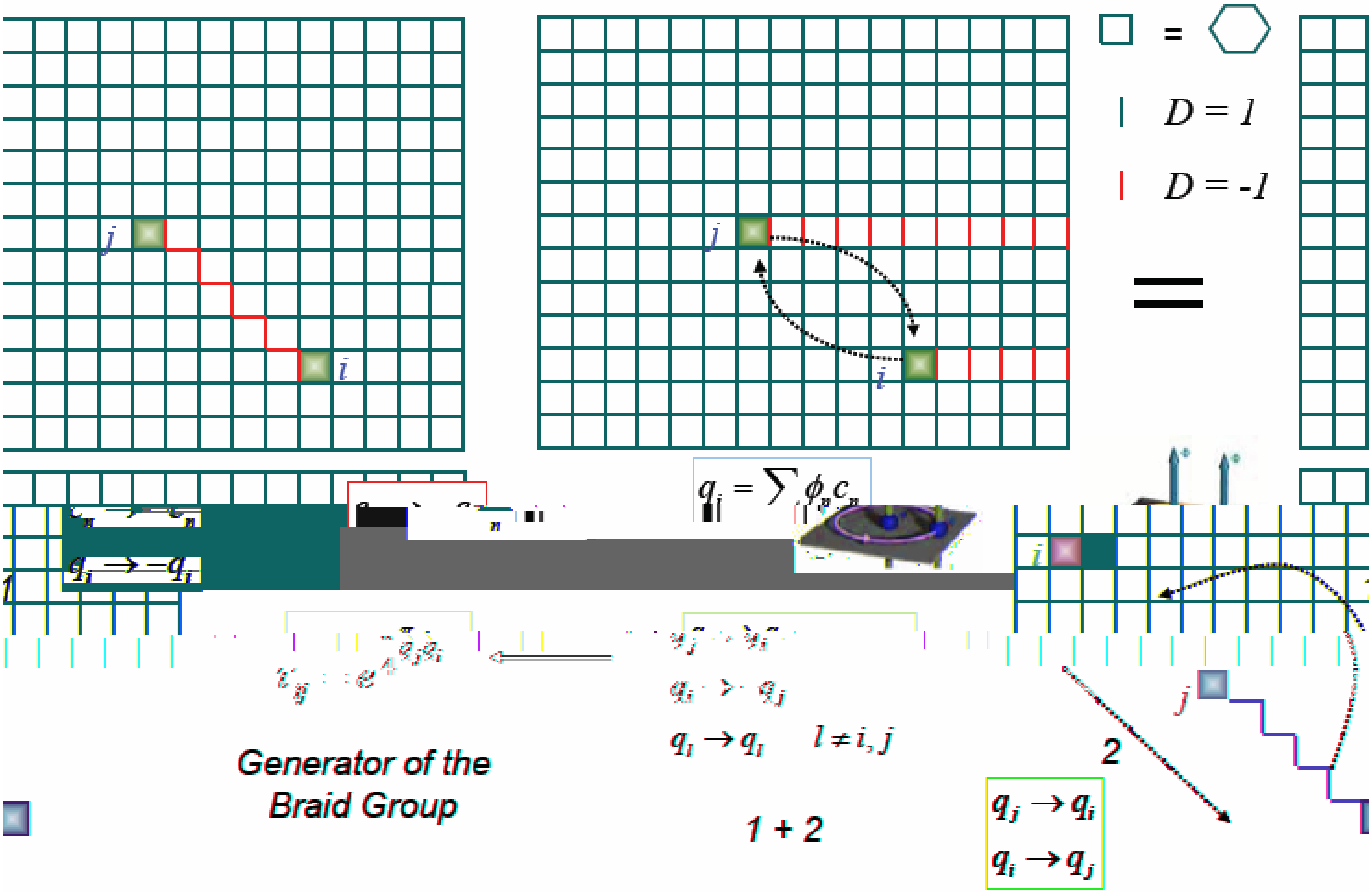
Edge Current and Edge Modes



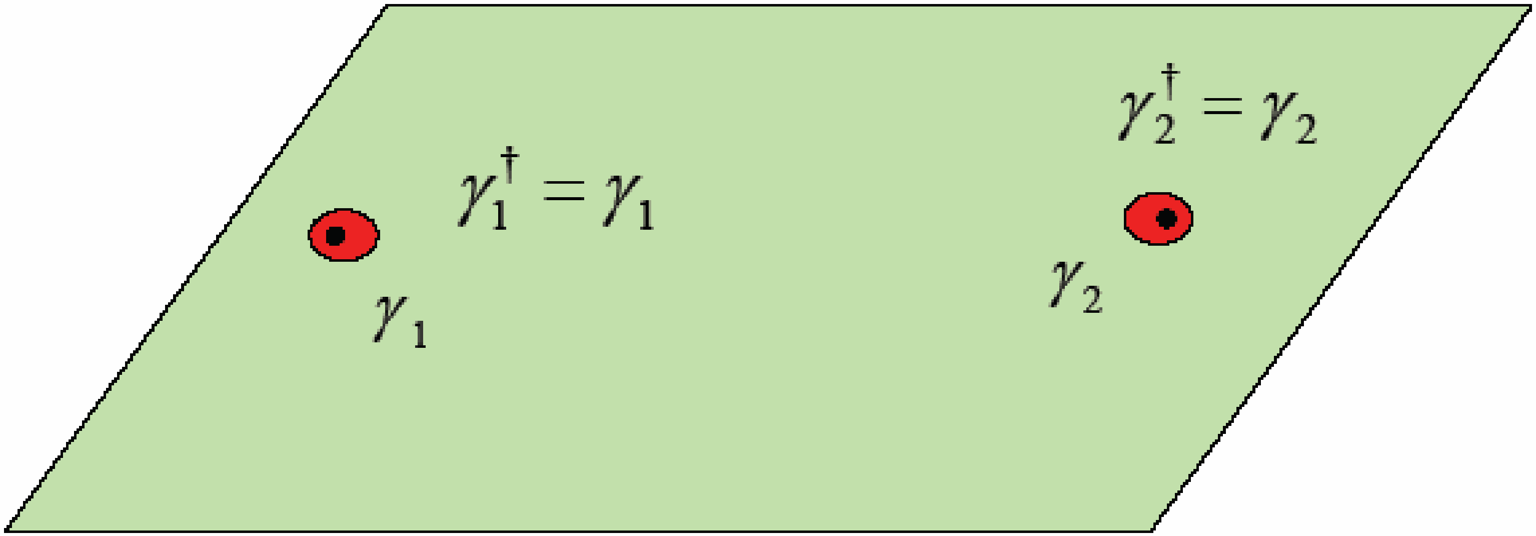
Edge Soliton: charge fractionalization



Non-Abelian anyons



Majorana Modes : Non-Locality



$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

A single γ mode cannot accommodate an electron!

Construct $\quad c = \gamma_1 + i\gamma_2$

$$c^\dagger = \gamma_1 - i\gamma_2$$

c's can be occupied by electrons

Non-local occupation

Majorana Modes : GS Degeneracy I

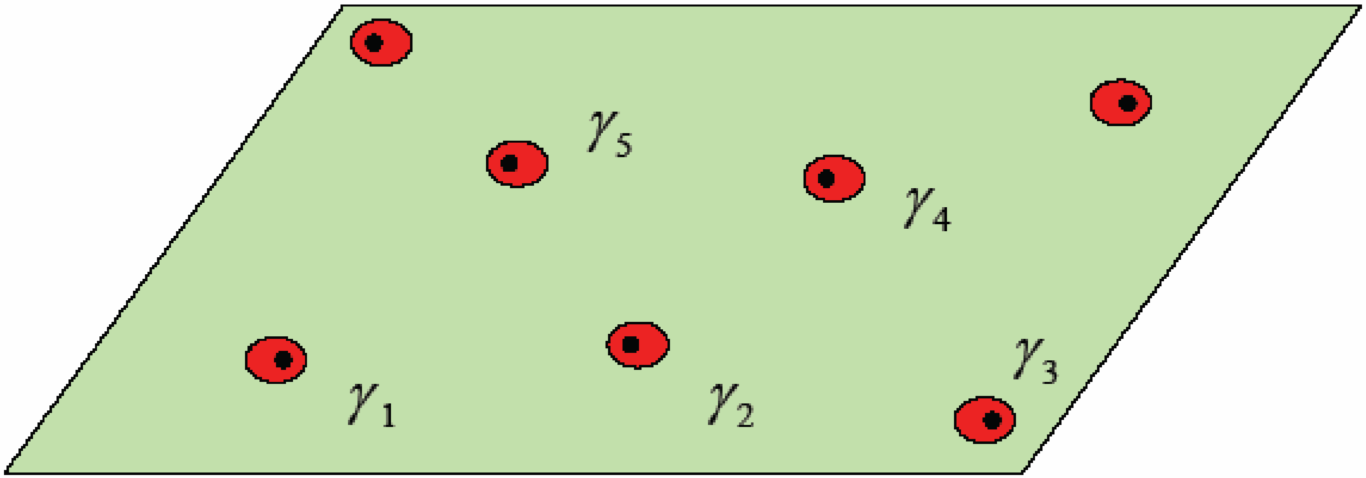
A pair of vortices support an electronic mode **at zero energy**

This mode can be unoccupied ($|0>$), or occupied ($|1>$)

Two states of a qubit

<u>The states are degenerate and completely non-local</u>

Majorana Modes : GS Degeneracy II



Consider 2n vortices / Majorana fermions => n electronic modes

They can be occupied / unoccupied by a SC QP

2^n -fold degenerate ground states protected by a gap $\omega_0 = \Delta_0^2 / E_F$

Any state in the GS manifold is a linear combination of them

Application to TQC

Non-local qubit + non-Abelian statistics (unitary operators)

Construct gates by braiding one vortex around another

Noiseless quantum computation

Conclusions

- The Kitaev model is a free Majorana fermion model with local Ising-like gauge field *without* redundant degrees of freedom.

- Topological ordering and quantum phase transitions can be characterized by non-local string order parameters. In the dual space, these string order parameters become local order parameters.

- The low-energy critical modes are Majorana fermions, not Goldstone bosons.

- Topological vortex excitations can be Abelian anyons or non-Abelian anyons.

Thank you very much for attention.