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Stochastic phase-space methods have many applications for carrying out quantum simulations. Some previous results:

Exact solutions for quantum optical time crystals.

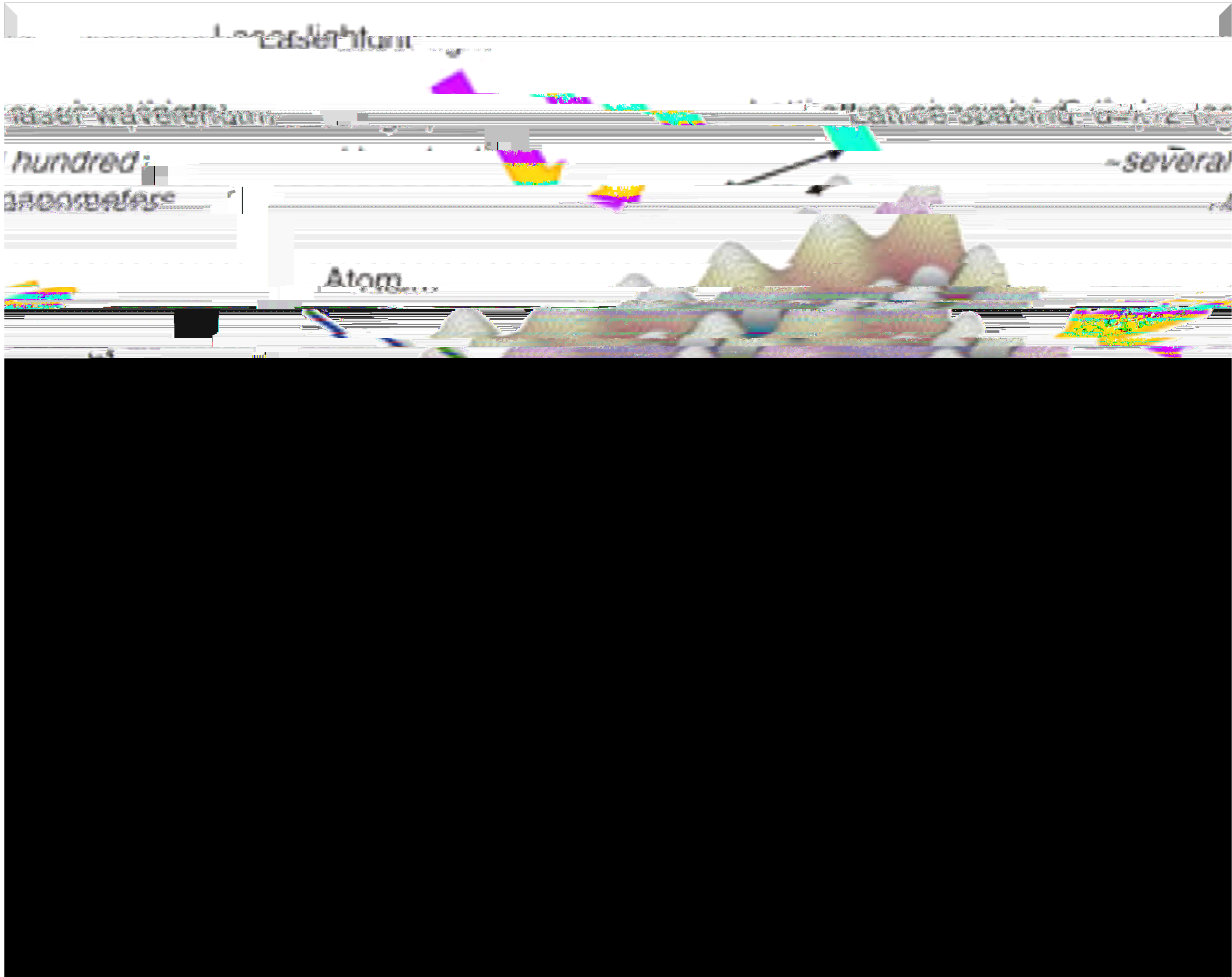
Dynamical simulations of million-mode interacting BECs.

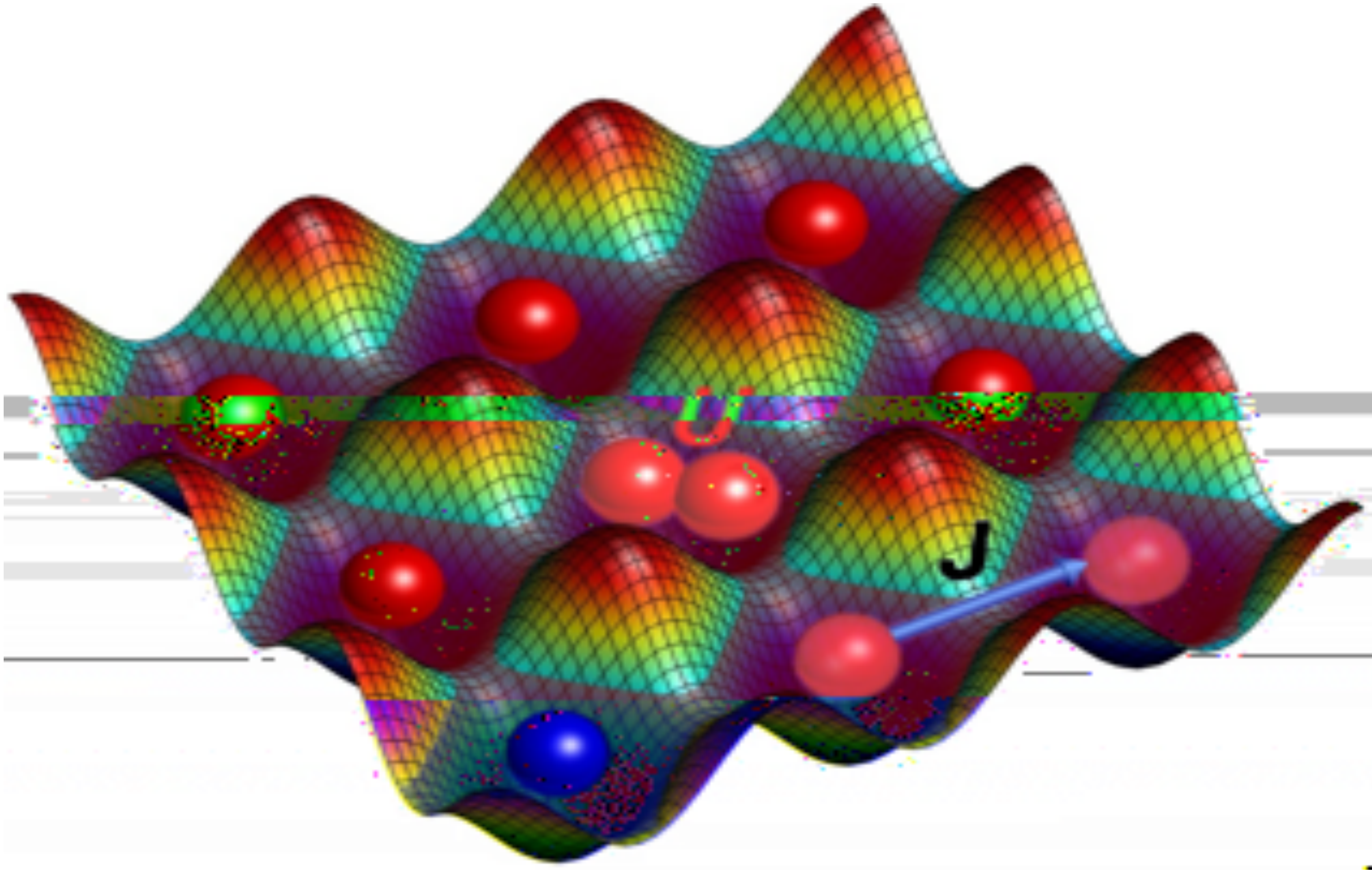
Simulations of optomechanical entanglement.

Fermionic problems are even more challenging: there is usually no means to sample the density matrix probabilistically. In this talk some recent advances for **fermions** will be treated, including

- A generalized Q and P-function definition applied to fermions
- Identities for mapping fermionic operators
- Examples: finite temperature shock waves, collective modes

What about many-body fermion and majorana systems?

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Suppose we have a positive definite, hermitian operator basis $\hat{\Lambda}\left(\vec{\lambda}\right)$ defined in a Hilbert space $\mathcal{H}$ of quantum mechanical operators, where $\vec{\lambda}$ is a vector in the phase-space domain $\mathcal{D}$. A generalized Q-function is defined as the inner product of the density matrix $\hat{\rho}$ with the operator basis:

$$Q\left(\vec{\lambda}\right) = \mathrm{Tr}\left[\hat{\Lambda}\left(\vec{\lambda}\right)\hat{\rho}\right].$$

Physical interpretation: this is simply the probability of observing the system in the state $\hat{\Lambda}\left(\vec{\lambda}\right)$.

Suppose we have an operator basis $\hat{\Lambda}\left(\vec{\lambda}\right)$ defined in a Hilbert space $\mathcal{H}$ of quantum mechanical operators, where $\vec{\lambda}$ is a vector in the phase space domain $\mathcal{D}$.

We can write the density matrix $\hat{\rho}$ using the operator basis:

$$\hat{\rho} = \int_{\mathcal{D}} P\left(\vec{\lambda}\right)\hat{\Lambda}\left(\vec{\lambda}\right) d\mu\left(\vec{\lambda}\right).$$

Physical interpretation: this is simply the expansion of the density matrix in terms of states $\hat{\Lambda}\left(\vec{\lambda}\right)$.

- This does not require hermiticity or positivity of $\hat{\Lambda}\left(\vec{\lambda}\right)$, and
- Generally different to Q owing to non-orthogonality; can use a different domain

We require the following completeness property: the identity operator $\hat{I}$ of the Hilbert space can be resolved as an integral over the phase-space, so that

$$\int_{\mathcal{D}} \hat{\Lambda}\left(\vec{\lambda}\right) d\mu\left(\vec{\lambda}\right) = \hat{I}\,.$$

This is called a resolution of unity.

Here $d\mu\left(\vec{\lambda}\right)$ is an associated integration measure on the phase-space.

- A set of identities that allows all operator moments of physical interest $\hat{O}_n$ to be mapped into differential operators is required, so that:

$$\hat{O}_n \hat{\Lambda}\left(\vec{\lambda}\right) = \mathcal{D}_n\left(\partial_{\vec{\lambda}}, \vec{\lambda}\right)\hat{\Lambda}\left(\vec{\lambda}\right)$$

[illegible] observable in the form of an [illegible] using the resolution of unity, an[illegible] operator moment can be represented as:

$$[illegible]$$

Here we consider normally-ordered Gaussian operators, with unit trace:

$$\hat{\Lambda}\left(\underline{\underline{\sigma}}\right) = \sqrt{\det\left[i\underline{\underline{\sigma}}\right]} : \exp\left[-\underline{\hat{a}}^{\dagger}\left(\underline{\underline{\sigma}}^{-1} - 2\underline{\underline{\bar{I}}}\right)\underline{\hat{a}}/2\right] : ,$$

with:

$$\underline{\underline{\bar{I}}} = \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \text{and} \quad \underline{\underline{\sigma}} = \begin{bmatrix} \mathbf{n}^T - \mathbf{I} & \mathbf{m} \\ -\mathbf{m}^* & \mathbf{I} - \mathbf{n} \end{bmatrix}.$$

The $2M \times 2M$ matrix $\underline{\underline{\sigma}}$ is the covariance matrix expressed in terms of an $M \times M$ hermitian matrix $\mathbf{n}$ and a compl antisymmetric $M \times M$ matrix $\mathbf{m}$.

We define a "stretched" variable $\underline{\underline{\zeta}}$ as:

$$\underline{\underline{\zeta}} = \underline{\underline{\bar{I}}} - 2\underline{\underline{\sigma}} = \underline{\underline{\tilde{\sigma}}} - \underline{\underline{\sigma}}.$$

We also define a normalized Gaussian basis $\hat{A}^N$, which in terms of these variables is:

$$\hat{A}^N\left(\underline{\underline{\zeta}}\right) = \frac{1}{N}\hat{A}\left(\frac{1}{2}\left[\underline{\underline{\bar{I}}} - \underline{\underline{\zeta}}\right]\right) S\left(\underline{\underline{\zeta}}^2\right).$$

$S\left(\underline{\underline{\zeta}}^2\right)$ is an even, positive scaling function, invariant under ... unitary transformations. These ... symmetry introduced by Altland and Zirnbauer.

The Gaussian operator $\hat{\Lambda}^N(\underline{\zeta})$ are the basis for the fermionic Q-function. This is a positive hermitian basis … resolution of unity. We have proved the following resolution

$$\hat{I} = \int_{\mathcal{D}} d\zeta \hat{\Lambda}^N\left(\underline{\zeta}\right),$$

where $d\zeta$ is the Riemannian measure on the symmetric space.

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70'$0%MS%GS%8)401"4690$0#"B%!S%&S%&S,:)'$*01%)=%!254/.4%E%46B%T\UTR`%0TRX`QS%

The fermionic Q-function is defined for any density matrix, in terms of the Gaussian basis, as:

$$Q\left(\underline{\zeta}\right) = \mathrm{Tr}\left[\hat{\rho}\hat{\Lambda}^{N}\left(\underline{\zeta}\right)\right].$$

- The Gaussian operators and the density matrix are positive definite hence the Q-function is positive.
- From the resolution of unity, the Q-function is normalized to unity: $\int d\zeta Q\left(\underline{\zeta}\right) = 1$.
- This fermionic Q-function is a non-negative probability distribution, and it is normalized to unity. It is also defined for any density-matrix.

In the extended variables, observables can be expressed in normal, antinormal or nested ordering. We consider the antinormal form of the observables, which are given by:

$$\left\langle\left\{\hat{a}\hat{a}^{\dagger}\right\}\right\rangle = \mathrm{Tr}\left[\hat{\rho}\begin{pmatrix}\hat{a}\hat{a}^{\dagger} & \hat{a}\hat{a}^{T}\\ \hat{a}^{\dagger T}\hat{a}^{\dagger} & \left(\hat{a}^{\dagger T}\hat{a}^{T}\right)^{T}\end{pmatrix}\right].$$

Using the resolution of unity, the observables can be expressed

$$\left\langle\left\{\hat{a}_{\alpha}\hat{a}_{\beta}^{\dagger}\right\}\right\rangle = \mathrm{Tr}\left[\int \hat{a}_{\zeta}\hat{\rho}^{\dagger}\left\{\hat{a}_{\alpha}^{N}:\hat{a}_{\beta}^{N}\right\}\right]$$

We prove the following differential identity

$$\Lambda^{N}\frac{\partial \ln S}{\partial \sigma}\underline{\underline{\sigma}}. \qquad \left\{\hat{a}:\hat{a}^{\dagger}\hat{\Lambda}^{N}:\right\} = -\underline{\underline{\sigma}}\hat{\Lambda}^{N} - \underline{\underline{\sigma}}\frac{\partial\hat{\Lambda}^{N}}{\partial\sigma}\underline{\underline{\sigma}} + \ldots$$

the normalization function ... Considering the explicit form for t...

servables are given by: ... S, in the limit $S \to 1$, the observa...

$$\ldots\left(\underline{\underline{\zeta}}\right) d\zeta - \frac{1}{2}\underline{\underline{I}}, \qquad \left\langle\left\{\underline{\hat{a}\hat{a}^{\dagger}}\right\}\right\rangle = C_M \int_{\mathcal{V}} \underline{\underline{\zeta}} Q \ldots$$

$$C_M = 2M - 1/2 \ldots$$

Thermal Q-function

- The density matrix of a thermal state is:

$$\hat{\rho}_{th} = \tilde{n}_{th} : \exp\left[-\hat{a}^{\dagger}\left(2 - \tilde{n}_{th}^{-1}\right)\hat{a}\right] : = \hat{\Lambda}_{th}\left(\tilde{n}_{th}\right),$$

- In this case the Q-function is:

[redacted]

Next we consider Majorana operators, with:

$$\gamma_1 = a + a^\dagger$$

$$\gamma_2 = i\left(a^\dagger - a\right)$$

t trace Gaussian opertor:

$$\hat{\Lambda}(\underline{x}) = N(\underline{x}) : \exp\left[i\hat{\gamma}^T \underline{\mathcal{I}} \left[\underline{I} - (\underline{\mathcal{I}}\,\underline{x} + \underline{I})^{-1}\right] \hat{\gamma}/2 \right] :$$

Here $N(\underline{x}) = \frac{1}{2^M}\sqrt{\det\left[\underline{\mathcal{I}} - \underline{x}\right]}$, $\underline{\mathcal{I}} = \begin{bmatrix} 0 & \underline{I} \\ -\underline{I} & 0 \end{bmatrix}$ and $\underline{I}$ is the $2M \times 2M$ identity matrix.

[illegible]

se the unordered differential identities derived from those We us

given previously to obtain the observable function $O(\underline{x})$. We

consider the correlation function $\hat{X}_{\mu\nu}$ given by:

$$\hat{X}_{\mu\nu} \equiv \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right].$$

$$\left\langle \underline{\underline{\widehat{X}}} \right\rangle = \int \underline{\underline{x}} P\left(\underline{\underline{x}}\right) d\underline{\underline{x}},$$

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8S%8S%:)4"-2B%7S%MS%GS%8)401"469b$0#"B%!S%&S%&SB%!254/.01%8"D/"<%E%98B%RX`Y`Z%0TRXZQS%

Classical domains of E. Cartan

1. The domain $\mathcal{R}_I$ of $m \times n$ complex matrices with: $\underline{\underline{I}}^{(m)} - \underline{\underline{Z}}\underline{\underline{Z}}^\dagger > 0$.
2. The domain $\mathcal{R}_{II}$ of $n \times n$ symmetric complex matrices with: $\underline{\underline{I}} - \underline{\underline{Z}}\underline{\underline{Z}}^* > 0$.
3. The domain $\mathcal{R}_{III}$ of $n \times n$ skew-symmetric (anti-symmetric) complex matrices with: $\underline{\underline{I}} + \underline{\underline{Z}}\underline{\underline{Z}}^* > 0$.

$z = (z_1, z_2, \ldots, z_n)$, where z_k are complex numbers,

M-phase space

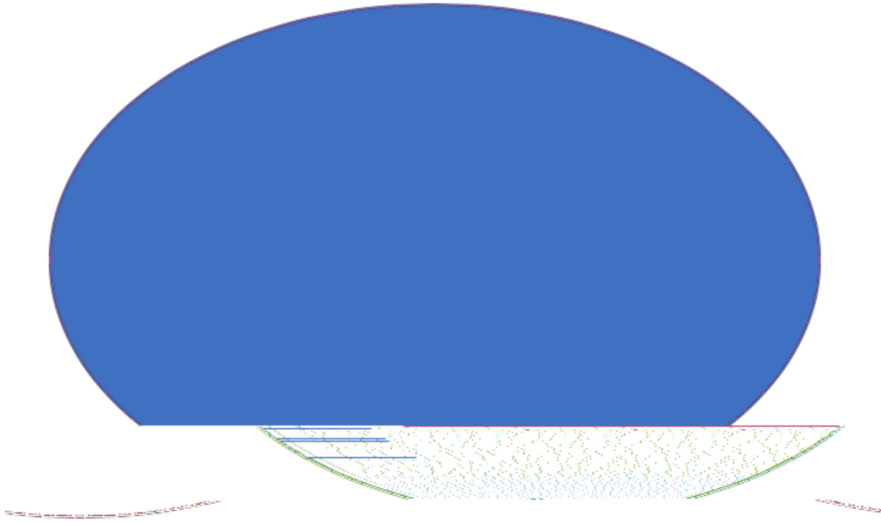
This is a REAL antisymmetric phase space:

$$\underline{\underline{x}} = \begin{bmatrix} i\left(\mathbf{n}^- + \mathbf{m}^-\right) & \mathbf{n}^+ + \mathbf{m}^+ - \mathbf{I} \\ -\mathbf{n}^+ + \mathbf{m}^+ + \mathbf{I} & i\left(\mathbf{n}^- - \mathbf{m}^-\right) \end{bmatrix}$$

where $\mathbf{n}^\pm = \mathbf{n} \pm \mathbf{n}^T$, $\mathbf{m}^\pm = \mathbf{m} \pm \mathbf{m}^*$ and $n_{ij} = \left\langle \hat{a}_i^\dagger \hat{a}_j \right\rangle$, $m_{ij} = \langle \hat{a}_i \hat{a}_j \rangle$.

$$\underline{\underline{x}}\underline{\underline{x}}^T - \underline{\underline{I}} < 0$$

Bounded real domain in
M(2M-1) = 1,6,15,28,.. dimensions:



Hamiltonian

$$\widehat{H} = \hbar\omega_{ij}\widehat{a}_i^\dagger\widehat{a}_j,$$

If we define the Majorana commutator as previously

$$\underline{\underline{\Omega}} = \begin{bmatrix} 0 & \boldsymbol{\omega} \\ -\boldsymbol{\omega} & 0 \end{bmatrix},$$

Majorana [illegible] we can re-express the Hamiltonian in terms of operators as

$$\hat{H} = \frac{\hbar}{2}\Omega_{\mu\nu}\hat{X}_{\mu\nu}.$$

Time-evolution

$$\ldots Q)\Big] .$$

$$\frac{dQ\left(\underline{x}\right)}{dt} = \Omega_{\mu\nu}\left[\frac{d}{dx_{\nu\kappa}}\left(x_{\mu\kappa}Q\right) - \frac{d}{dx_{\kappa\mu}}\left(x_{\kappa\nu} \ldots\right.$$

tic

This leads to a characteristic equation for all stochas... trajectories (P or Q):

$$\frac{d\underline{x}}{dt} = \left[\underline{\Omega}, \underline{x}\right] .$$

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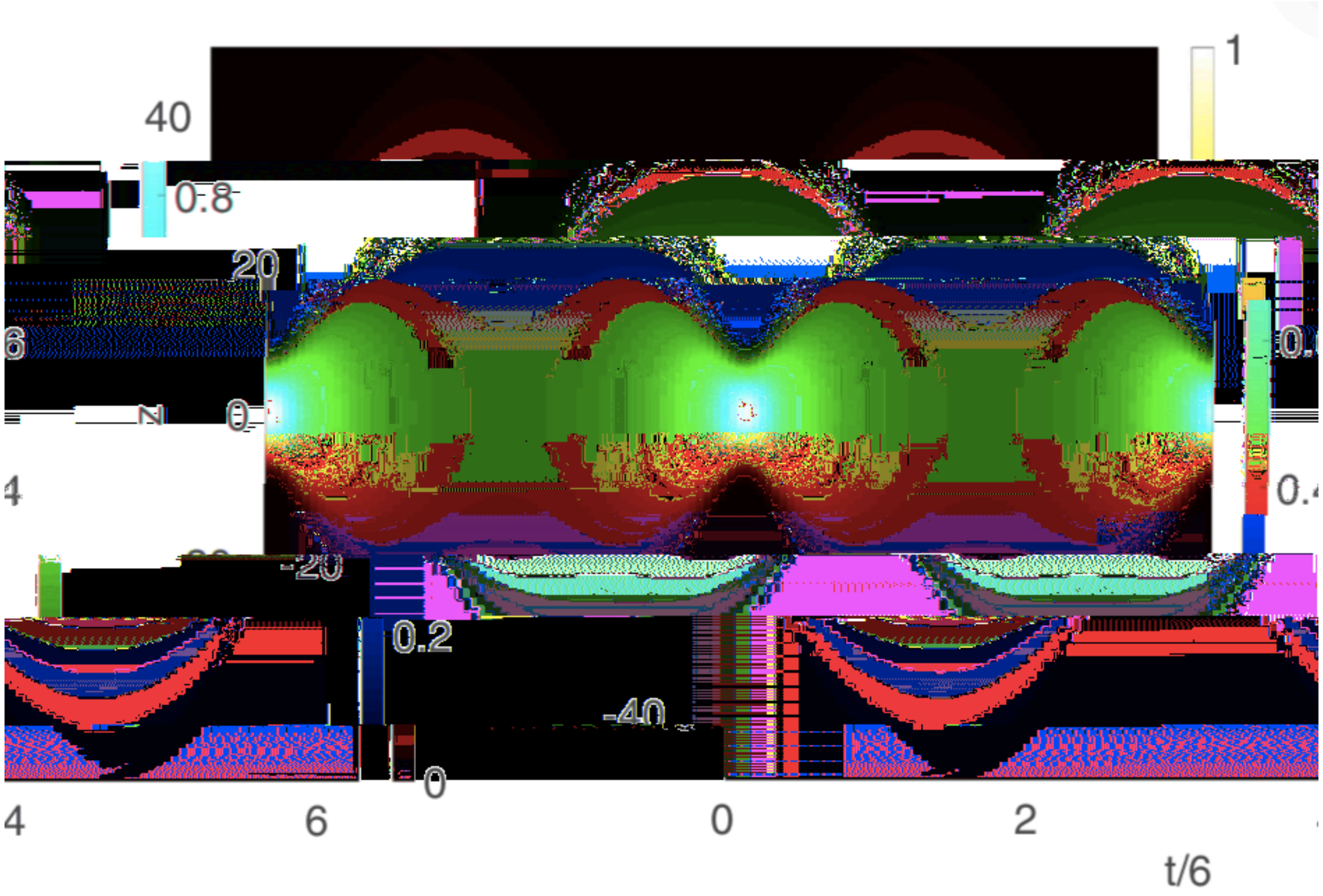
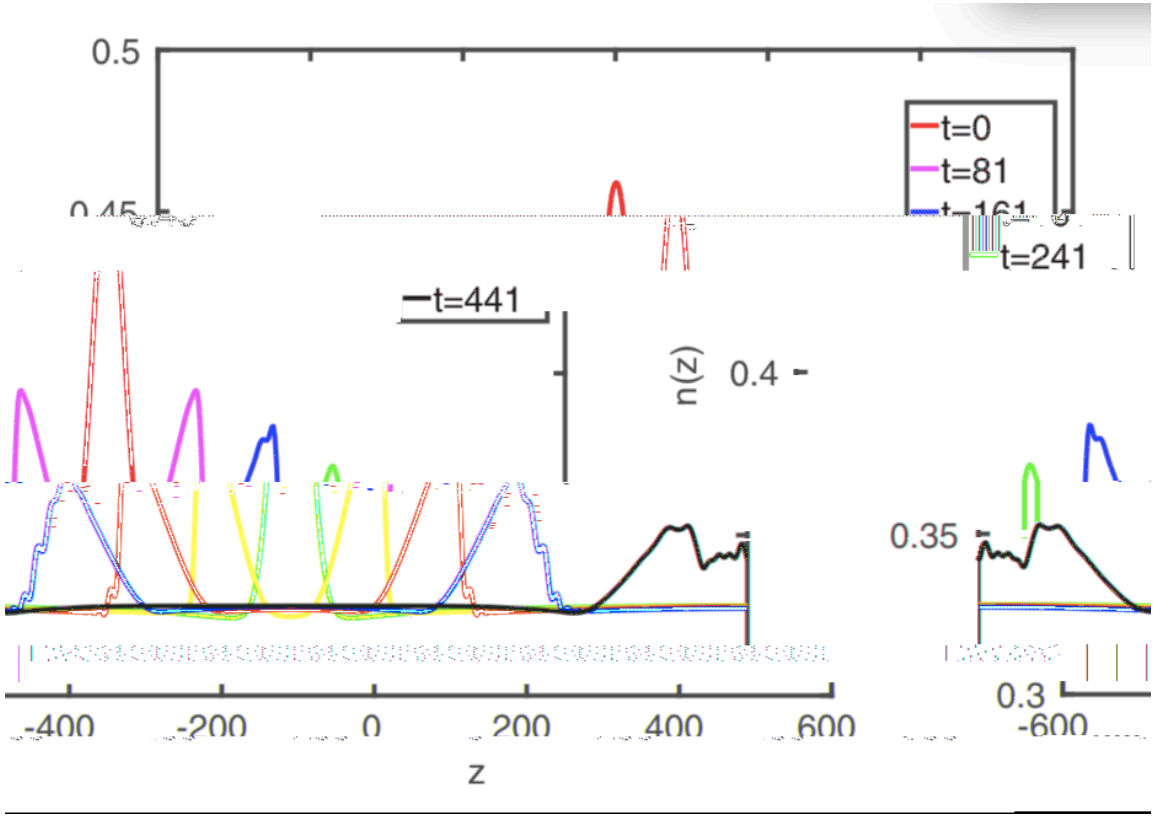
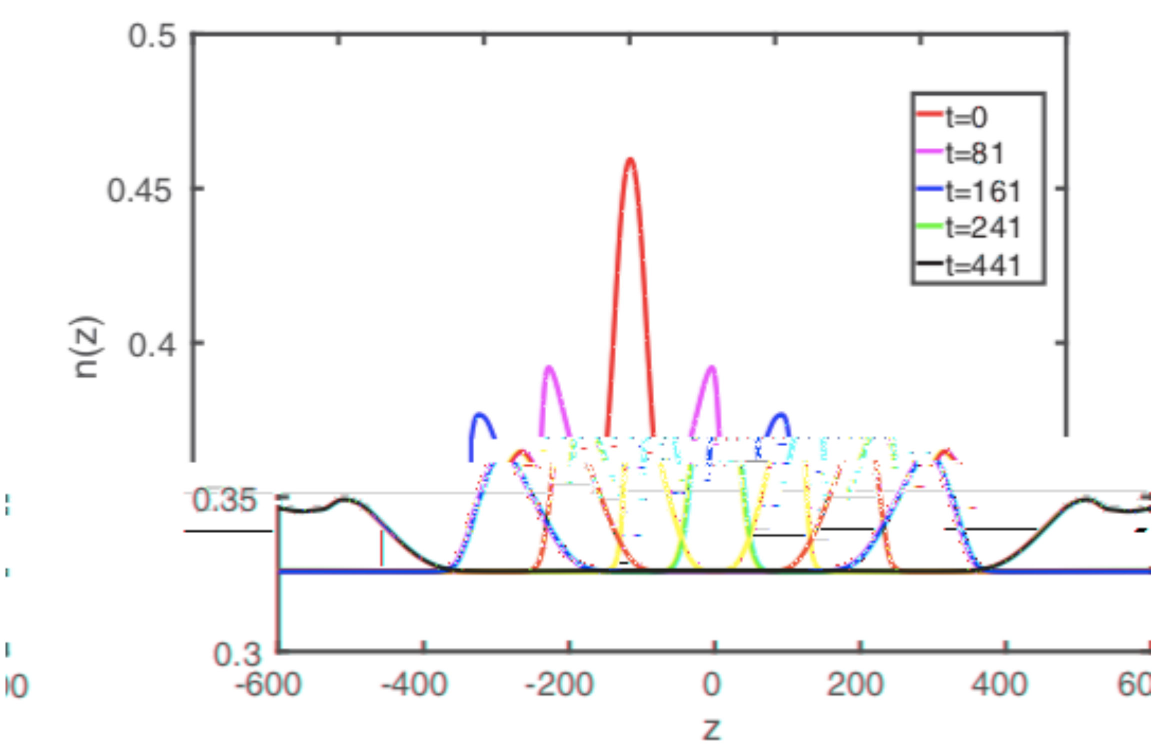



n(z)
z
t=0
t=81
t=241
t=441
0.5
0.4
0.35
0.3
-600
-400
-200
0
200
400
600


n(z)
z
t=0
t=81
t=161
t=241
t=441
0.5
0.45
0.4
0.35
0.3
-600
-400
-200
0
200
400

Problems with quantum measurement

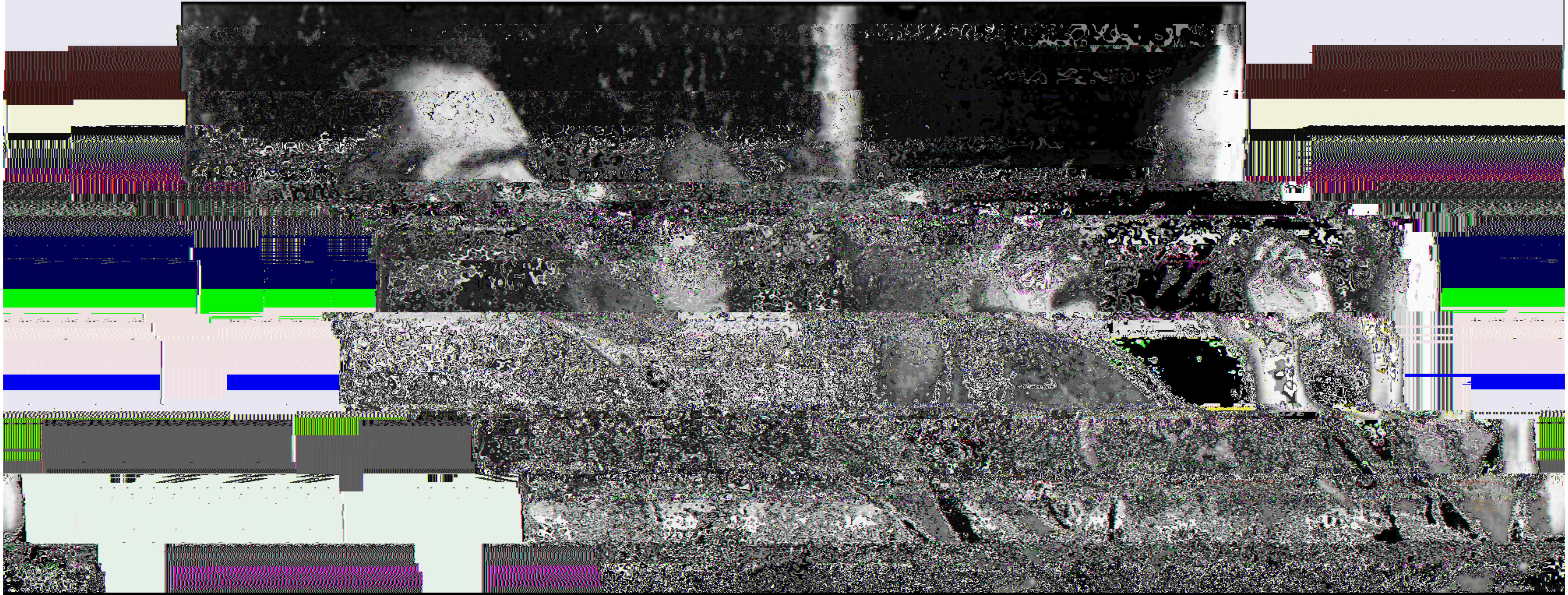
projection

- ...d?
- ...physics?
- ...part of the universe?
- ...quantum cosmology?
- ...ot solved through decoherence
- ...t project one eigenvalue

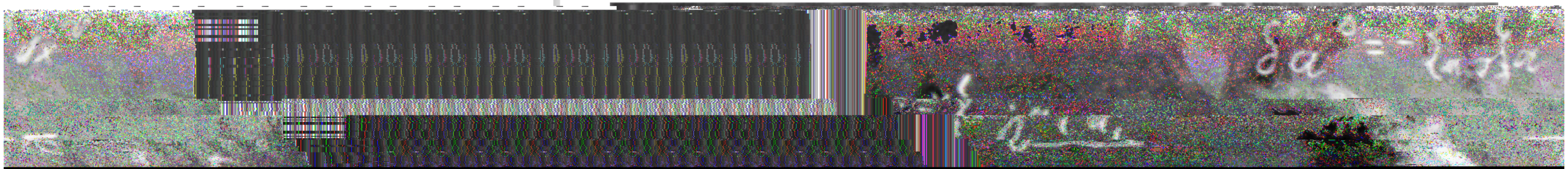
Measurement is regarded as a non-unitary

- But how is a measurement defined
- Why is it different to unitary
- What if the observer is
- How do we treat
- Problem is no
- This does not

Quantum measurement discussed by Einstein, Bohr



What did Einstein believe?



What about Bohr?

Bohr's ideas a

- Operational measurem

n! These goals are not in contradiction

ons

ne

measurement

l knowledge

of the universe

Ideally a physical theory should be

- **Objective:** external entities exist
- **Measurable:** through physical operati
- **Local:** objects are localized in spacetim
- **Consistent:** no special treatment of m
- **Stochastic:** we only have partia
- **Unified:** observers are part c

Q-functions

…bjective models exist

$$Q(\lambda, t) = Tr\left\{\hat{\Lambda}(\lambda)\hat{\rho}(t)\right\},$$

…e quantum density matrix,

…$\Pi_{b,f}\hat{\Lambda}_b(\psi)\hat{\Lambda}_f(x)$ is a positive-definite operator basis,

…oint in the phase-space.

…ve an expansion of the Hilbert space identity operator $\hat{I}$,

…n an integration measure $d\boldsymbol{\lambda}$,

- $\hat{I} = \int \hat{\Lambda}(\lambda)\, d\lambda$.

The Q-function proves o…

$Q(\boldsymbol{\lambda}, t)$…

- where $\hat{\rho}(t)$ is th…
- $\hat{\Lambda}(\lambda) = \Pi$…
- $\boldsymbol{\lambda}$ is a p…

This must giv…
so that, give…

Quantum fields

$N-$component bosonic field $\hat{\psi}(r)$

- Defined with a space-time coordinate r, where

$r = (r^1, \ldots r^d) = (\mathbf{r}, t)$.

M/N modes.

- The indices i include
- N internal degrees of freedom

Projection operators

ector,

state

-time

Bosonic case

For bosonic fields, $\hat{\Lambda}$ is proportional to a coherent state proje

$$\hat{\Lambda}(\boldsymbol{\alpha}) \equiv |\boldsymbol{\alpha}\rangle_c \langle \boldsymbol{\alpha}|_c / \pi^M .$$

The state $|\boldsymbol{\alpha}\rangle_c$ is a normalized Bargmann-Glauber coherent s

with $\hat{a}_j |\boldsymbol{\alpha}\rangle_c = \alpha_j |\boldsymbol{\alpha}\rangle_c$ and

$$\hat{\boldsymbol{\psi}}(x) |\boldsymbol{\alpha}\rangle_c = \boldsymbol{\psi}(x) |\boldsymbol{\alpha}\rangle_c ,$$

Hence, phase-space is composed of local fields in space

$$\lambda \to \psi(x)$$

This is a probabilistic representation

- $\int Q(\lambda)\, d\lambda = 1$
- $Q(\lambda) \geq 0$

...a possible universe

- ...elds in space time
- ...universe.
- ...objective universe

Every phase-space coordinate is ...

- Each $\lambda(t)$ is a set of fie...
- Every $\lambda(t)$ is a possible...
- There is only one...

Observables

We can compute observables in the usual way

Quantum expectations $\left\langle \hat{O} \right\rangle_Q$ of ordered observables $\hat{O}$ are identical to classical probabilistic averages $\left\langle O \right\rangle_C$ including corrections for operator re-ordering if necessary - so that:

$$\left\langle \hat{O} \right\rangle_Q = \left\langle O \right\rangle_C = \int d\lambda\, O(\lambda) P(\lambda)$$

We can also add a model of the measurement, which models the growth of the observable to a macroscopic size.

Differential equations from operator mappings

...k equation

...tion is of the form:

$$\dot{Q} = \left[-\partial_\mu A_\mu + \frac{1}{2}\partial_\mu \partial_\nu D_{\mu\nu}\right] Q$$

...ver, D is not positive definite, and in fact $Tr(D) = 0$, yet Q is positive definite. This is because it corresponds to a simultaneous positive and negative time evolution. This has boundary conditions in the past and the future, and has a real action principle.

Operator correspondences → Fokker-Plan...

Fokker-Planck equa...

However, D...

positi...

Amplification is fundamental to measurement!

From the amplified macroscopic value X, with gain G the experimentalist *infers* an eigenvalue of $\tilde{X} = X_0 + \varepsilon / G$, with a probability distribution of

$$P\left(\tilde{X}\right) = \left(G / \sqrt{2\pi}\right) exp\left(-G^2\left(\tilde{X} - X_0 \ldots\right.\right.$$

$\ldots)/2)\ldots$

- Vacuum fluctuations relative… ly negligible at large gain, allowing eigenvalue meas… surement.

Conclusions

...s a fundamental quantum theory

- ...in exact quantum equations
- ...abilistic
- ...quirement to collapse wave function

Q function as...

- Can obta...
  - Proba...
  - No re...